On Sequence of Functions Involving I-function

Nawal Kishor Jangid
Department of Mathematics,
Swami Keshvanand Institute of Technology, Management and Gramothan, Jaipur, India
Email: jangid.0008@gmail.com

Received 30 January 2017, received in revised form 04 March 2017, accepted 10 March 2017

Abstract: **In the present paper we introduce a sequence of functions involving Rathi's I-function [6] by using operational techniques. Three generating relations pertaing to this sequence of functions have also been established. These generating relations are unified in nature and act as key formulae from which a large number of known or new results can be obtained as special cases. For the sake of illustration, some special cases of our main findings have been recorded here.**

Key words :- Special function, generating relations, I-function, sequence of functions

1. INTRODUCTION

Rathie [6] has introduced the I-function which is a new generalization of the $\overline{H}$-function which was the generalization of the familiar H-function of Fox [8]. The I-function will be defined and represented by the following Mellin-Barnes type contour integral [6].

$$I_{P,Q}^{M,N}\left[z\right]=I_{P,Q}^{M,N}\left[z\left|\begin{matrix}\left(a_j,A_j;\alpha_j\right)_{1,N},\left(a_j,A_j;\alpha_j\right)_{N+1,P}\\\left(b_j,B_j;\beta_j\right)_{1,M},\left(b_j,B_j;\beta_j\right)_{M+1,Q}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L\theta\left(s\right)z^s ds, \qquad (1.1)$$

where $\theta\left(s\right)$ is given by

$$\theta\left(s\right)=\frac{\prod_{j=1}^{M}\Gamma^{\beta_j}\left(b_j-B_js\right)\prod_{j=1}^{N}\Gamma^{\alpha_j}\left(1-a_j+A_js\right)}{\prod_{j=M+1}^{Q}\Gamma^{\beta_j}\left(1-b_j+B_js\right)\prod_{j=N+1}^{P}\Gamma^{\alpha_j}\left(a_j-A_js\right)} \qquad (1.2)$$

It may be noted that the (θ)*s* contains fractional powers of the gamma function and M,N,P,Q are integers satisfying $0\le M\le Q, 0\le N\le P.$ A_j, j = 1,2,…,P; B_j, j = 1,2,…,Q ; α_j, j = 1,2,…,P and β_j, j = 1,2,…,Q are positive real numbers. a_j, j = 1,2,…,P and b_j, j = 1,2,…,Q are complex numbers.

For the nature of contour L, sufficient conditions of convergence of defining integral (1.1) and other details about the I-function one may refer to [6].

2. PREVIOUS WORK

We also recall here the following known functions and earlier works :

The Rodrigues formula for generalized Lagurre polynomials is given by Mittal [2] as follows :

$$L_{kn}^{(\alpha)}(x)$$

$$=\frac{1}{n!}x^{-\alpha}\exp(p_k(x))\ D^n[x^{\alpha+n}\exp(-p_k(x))] \qquad (2.1)$$

where $p_k(x)$ is a polynomial in x of degree k.

The relation (2.1) can also be proved by Mittal [3] as

$$L_{kn}^{(\alpha+s-1)}(x)\ =\frac{1}{n!}x^{-\alpha-n}\exp(p_k(x))$$
$$L_s^n[x^{\alpha}\exp(-p_k(x))] \qquad (2.2)$$

where s is constant and $L_s=x(s+xD)$.

A sequence of functions $W_n^{(\alpha)}(x;a,k,s)$ is studied by Srivastava and Singh [7] and is defined by:

$$W_n^{(\alpha)}(x;a,k,s)$$
$$=\frac{x^{-\alpha}}{n!}\exp\left\{p_k(x)\right\}\theta^n\left[x^{\alpha}\exp\left\{-p_k(x)\right\}\right] \qquad (2.3)$$

where $\theta\ \equiv\ x^a(s+xD)$ and s is constant and $p_k(x)$ is a polynomial in x of degree k.

We also defined a new sequence of function $\left\{W_n^{(M,N;P,Q,\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)\right\}_{n=0}^{\infty}$ involving Rathi's I-function in the following manner:

$$W_n^{(M,N;P,Q,\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$
$$=\frac{1}{n!}x^{-\alpha}I_{P,Q}^{M,N}\{p_k(x)\}\left(T_x^{a,s}\right)^n\left[x^{\alpha}I_{P,Q}^{M,N}\{-p_k(x)\}\right] \tag{2.4}$$

where $T_x^{a,s}\equiv x^a(s+xD), D\equiv\frac{d}{dx}$, a and s are constants, $\beta\geq 0,$ k is finite and non-negative integer, $p_k(x)$ is a polynomial in x of degree k.

Some generating relations and finite summation formulae of class of polynomials or sequence of functions have been obtained by using the properties of the differential operators, $T_x^{a,s}\equiv x^a(s+xD), T_x^{a,1}\equiv x^{a,1}(1+xD),$ where $D=\frac{d}{dx}$, is based on the work of Mittal [4], Patil and Thakare [5], Srivastava and Singh [7].

Some useful operational techniques are as follows :

$$\exp\left(tT_x^{a,s}\right)\left(x^{\beta}f(x)\right)$$
$$=x^{\beta}(1-ax^at)^{-\left(\frac{\beta+s}{a}\right)}f\left(x(1-ax^at)^{-1/a}\right) \tag{2.5}$$

$$\exp\left(tT_x^{a,s}\right)\left(x^{\alpha-an}f(x)\right)$$
$$=x^{\alpha}(1+at)^{-1+\left(\frac{\alpha+s}{a}\right)}f\left(x(1+at)^{1/a}\right) \tag{2.6}$$

$$\left(T_x^{a,s}\right)^n(xuv)=x\sum_{m=0}^{\infty}\binom{n}{m}\left(T_x^{a,s}\right)^{n-m}(v)\left(T_x^{a,1}\right)^m(u) \tag{2.7}$$

$$(1+xD)(1+a+xD)(1+2a+xD)$$
$$...(1+(m-1)D+xD)x^{\beta-1}=a^m\left(\frac{\beta}{a}\right)_m x^{\beta-1} \tag{2.8}$$

$$(1-at)^{\frac{-\alpha}{a}}=(1-at)^{\frac{-\beta}{\alpha}}\sum_{m=0}^{\infty}\left(\frac{\alpha-\beta}{a}\right)_m\frac{(at)^m}{m!} \tag{2.9}$$

3. GENERATING RELATIONS

In this section of the present paper we discuss the following three generating relations as follows:

First Generating Relation :

$$\sum_{n=0}^{\infty}W_n^{(M,N;P,Q;\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$
$$\times x^{-an}t^n=(1-at)^{-\left(\frac{\alpha+s}{a}\right)}I_{P,Q}^{M,N}(p_k(x))$$
$$I_{P,Q}^{M,N}(-p_k(x(1-at)^{-1/a})) \tag{3.1}$$

Proof :

To prove the above generating relation let us consider from equation (2.4)

$$\sum_{n=0}^{\infty}W_n^{(M,N;P,Q;\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$
$$\times t^n=x^{-\alpha}I_{P,Q}^{M,N}(p_k(x))\exp(tT_x^{a,s})$$
$$\left[x^{\alpha}I_{P,Q}^{M,N}(-p_k(x))\right] \tag{3.2}$$

Now using (2.6) in relation(3.5), it will reduce to

$$\sum_{n=0}^{\infty}W_n^{(M,N;P,Q;\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$
$$\times t^n=x^{-\alpha}I_{P,Q}^{M,N}(p_k(x))x^{\alpha}(1-ax^at)^{-\left(\frac{\alpha+s}{a}\right)}$$
$$I_{P,Q}^{M,N}(-p_k(x(1-ax^at)^{-1/a})) \tag{3.3}$$

Thereafter replacing t by tx^{-a} we get the desired result.

Second Generating Relation :

$$\sum_{n=0}^{\infty}W_n^{(M,N;P,Q;\alpha-an)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$
$$\times x^{-an}t^n=(1+at)^{-1+\left(\frac{\alpha+s}{a}\right)}I_{P,Q}^{M,N}(p_k(x))$$
$$I_{P,Q}^{M,N}(-p_k(x(1+at)^{1/a})) \tag{3.4}$$

Proof :
To prove the above generating relation let us consider

$$\sum_{n=0}^{\infty} x^{-an} W_n^{(M,N;P,Q;\alpha-an)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$

$$\times t^n = x^{-\alpha} I_{P,Q}^{M,N}(p_k(x)) \exp(tT_x^{a,s})$$

$$\left[x^{\alpha-an} I_{P,Q}^{M,N}(-p_k(x))\right] \qquad (3.5)$$

Now using (2.6) in relation(3.5), it will reduce to

$$\sum_{n=0}^{\infty} x^{-an} W_n^{(M,N;P,Q;\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$

$$\times t^n = x^{-\alpha} I_{P,Q}^{M,N}(p_k(x)) x^{\alpha} (1+at)^{-1+\left(\frac{\alpha+s}{a}\right)}$$

$$I_{P,Q}^{M,N}(-p_k(x(1+at)^{1/a})) \qquad (3.6)$$

$$= I_{P,Q}^{M,N}(p_k(x))(1+at)^{-1+\left(\frac{\alpha+s}{a}\right)}$$

$$I_{P,Q}^{M,N}(-p_k(x(1+at)^{1/a}))$$

This completes the proof of second generating relation.

Third Generating Relation :

$$\sum_{n=0}^{\infty} \binom{m+n}{m} W_n^{(M,N;P,Q;\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$

$$\times t^n = (1-at)^{-\left(\frac{\alpha+s}{a}\right)} \frac{I_{P,Q}^{M,N}(p_k(x))}{I_{P,Q}^{M,N}(p_k(x(1-at)^{-1/a}))}$$

$$\times W_n^{(M,N;P,Q,\alpha)}(x(1-at)^{-1/a};a,k,s,$$

$$\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M) \qquad (3.7)$$

Proof :
To prove the above generating relation we rewrite the equation (2.4) as

$$(T_x^{a,s})^n \left[x^{\alpha} I_{P,Q}^{M,N}\{-p_k(x)\}\right] = n! x^{\alpha} \frac{1}{I_{P,Q}^{M,N}\{p_k(x)\}}$$

$$W_n^{(M,N;P,Q,\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M) \qquad (3.8)$$

or

$$\exp(tT_x^{a,s})\left\{(T_x^{a,s})^n \left[x^{\alpha} I_{P,Q}^{M,N}(-p_k(x))\right]\right\}$$

$$= n! \exp(tT_x^{a,\alpha}) \left[x^{\alpha} \frac{1}{I_{P,Q}^{M,N}\{p_k(x)\}} W_n^{(M,N;P,Q,\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M) \right] \qquad (3.9)$$

$$\sum_{n=0}^{\infty} \frac{t^m}{m!} (T_x^{a,s})^{m+n} \left[x^{\alpha} I_{P,Q}^{M,N}(-p_k(x))\right]$$

$$= n! \exp(tT_x^{a,s}) \left[x^{\alpha} \frac{1}{I_{P,Q}^{M,N}\{p_k(x)\}} W_n^{(M,N;P,Q,\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M) \right]$$

on using (2.5) , (3.9) reduces to

$$\sum_{n=0}^{\infty} \frac{t^m}{m!} (T_x^{a,s})^{m+n} \left[x^{\alpha} I_{P,Q}^{M,N}(-p_k(x))\right]$$

$$\sum_{m=0}^{\infty} \frac{t^m (m+n)!}{m! n!} x^{\alpha}$$

$$\frac{W_{m+n}^{(M,N;P,Q,\alpha)}(x(1-ax^a t)^{-1/a};a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)}{I_{P,Q}^{M,N}\{p_k(x)\}}$$

$$= n! x^{\alpha} (1-ax^a t)^{-\left(\frac{\alpha+s}{a}\right)}$$

$$\frac{W_n^{(M,N;P,Q,\alpha)}(x(1-ax^a t)^{-1/a};a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)}{I_{P,Q}^{M,N}\{p_k(x(1-ax^a t)^{-1/a})\}} \qquad (3.10)$$

Now using (3.8) we get

$$= x^{\alpha}(1-ax^{a}t)^{-\left(\frac{\alpha+s}{a}\right)} \frac{W_n^{(M,N;P,Q,\alpha)}(x(1-ax^{a}t)^{-1/a};a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)}{I_{P,Q}^{M,N}\left\{p_k(x(1-ax^{a}t)^{-1/a})\right\}}$$

(3.11)

Therefore we get

$$\sum_{m=0}^{\infty}\binom{m+n}{n} W_{m+n}^{(M,N;P,Q,\alpha)}(x;a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)t^m$$

$$= (1-ax^{a}t)^{-\left(\frac{\alpha+s}{a}\right)} \frac{I_{P,Q}^{M,N}\left\{p_k(x)\right\}}{I_{P,Q}^{M,N}\left\{p_k(x(1-ax^{a}t)^{-1/a})\right\}} W_n^{(M,N;P,Q,\alpha)}(x(1-ax^{a}t)^{-1/a};a,k,s,\alpha_{N+1},...,\alpha_P,\beta_1,...,\beta_M)$$

(3.12)

Finally by replacing t by tx^{-a}, we get the required result.

4. SPECIAL CASES

(i) By taking the exponents α_j=1($j=N$+1,...P) and β_j=1(j = 1,2,...P) in (3.1), (3.4) and (3.7) the I-function reduces to th H- function and we get the results obtained by Agrawal[1].

(ii) If we put A_j=B_j=1 in the results obtained in above special case (I) the $\overline{H}$-function reduces to Fox's $\overline{H}$-function and we get the results obtained by Agrawal[1].

5.CONCLUSION

In this paper, we discussed a new sequence of functions involving I-function by using operational techniques with the help of which we established some generating relations and also recorded some special cases involving $\overline{H}$-function. On account of most general nature of the $\overline{H}$-function large number of sequences and polynomials involving simpler functions can be easily obtained as their special cases.

REFERENCES

[1]. Agrawal, P. Chand, M. and Dwivedi, S., A study on new sequence of functions involving -function, American J. of Appl. Mathamatics and Statics, 2014, Vol. 2, No. 1, 34-39.

[2]. Mittal, H.B., A generalization of Laguerre polynomial, Publ. Math. Debrecen 1971, 18, 53-58.

[3]. Mittal, H.B., Operational representations for the generalized Laguerre polynomial, Glasnik Mat. Ser III 1971, 26(6), 45-53.

[4]. Mittal, H.B., Bilinear and Bilateral generating relations, American J. Math. 1977, 99, 23-45.

[5]. Patil, K. R. and Thakare, N.K., Operational formulas for a function defined by a generalized Rodrigues formula-II, Sci. J. Shivaji Univ. 1975, 15, 1-10.

[6]. Rathi, A.K., A new generalization of generalized hypergeometric function, Le Matematiche, 52(1997), 297-310.

[7]. Srivastava, A. N. and Singh, N. P., Some generating relations connected with a function defined by a generalized Rodrigues formula, Indian Pure Appl. Math. 1979, 10(10), 1312-1317.

[8]. Srivastava, H. M., Gupta, K. C. and Goyal, S. P., The H-function of one and tqo variables with applications, South Aian Publishers, New Delhi, Madras, 1982.

◆ ◆ ◆