On Infinite Series Involving Lucas Numbers

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Abstract: **In this paper the Binet form of Lucas numbers is solved using Geometric sequence. Moreover, in this paper, a closed form expressions for infinite series involving Lucas numbers are derived for finding the value of π.**

***Keywords*— Lucas number, Binet form, Inverse tangent function.**

1. INTRODUCTION

In 1202, the Italian mathematician Leonardo Fibonacci, introduced the Fibonacci sequence F_n using rabbit problem [1]. The Lucas numbers named after the French mathematician Francois Edouard Anatote Lucas in the beginning of the nineteenth century. The Lucas number sequence is also one of the most popular linear sequences in Mathematics. The Lucas sequence L_n is defined by recurrence relation,

$$L_n = L_{n-1} + L_{n-2},\ n > 1,\ L_0 = 2,\ L_1 = 1 \qquad \ldots (1.1)$$

It can also be represented in matrix form by,

$$\begin{pmatrix} L_{n-1} \\ L_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_{n-2} \\ L_{n-3} \end{pmatrix} \qquad \ldots (1.2)$$

The Lucas numbers are closely related to Fibonacci numbers. The two sequences are related by the formula,

$$F_n = \frac{1}{5}\left(L_{n-1} + L_{n+1}\right),\ n \ge 1 \qquad \ldots (1.3)$$

The Binet form for the Lucas sequence is

$$L_n = \alpha^n + \beta^n,\ n \ge 1 \qquad \ldots (1.4)$$

The proof of above Binet form is given by Datta [2] and many other authors also. Stakhov and Rozin [3] developed theory of Binet formulas for Fibonacci and Lucas p-numbers. Koshy [4] studied Applications of Fibonacci and Lucas Numbers. Hoggatt and Ruggles [5] produced some summation identities for Fibonacci and Lucas numbers involving the arctan function. Vajda [6] developed Theory and Applications of Fibonacci & Lucas numbers and their Golden Section.

In this paper the Binet formula for Lucas numbers is derived using Geometric sequence and we have extended a theorem found in [7] and a new closed- form evaluations of infinite series involving Lucas numbers are derived by elementary methods for finding the value of π .

2. PROOF OF BINET FORMULA

If we find the quotient of consecutive terms of Lucas numbers, it approaches to 1.618033… , called Golden Ratio. The irrational number, Golden ratio, is widely used in modern sciences. Many authors [8, 9, 10] have already worked on Golden Section. Here we are using this ratio to show that Lucas sequence, resembles to Geometric sequence. Geometric sequence has terms $G_n = a \cdot r^n$

We know,

$$L_n = L_{n-1} + L_{n-2}$$

$$\Rightarrow a \cdot r^n = a \cdot r^{n-1} + a \cdot r^{n-2}$$

$$\Rightarrow r^2 = r + 1$$

$$\Rightarrow r = \frac{1 \pm \sqrt{5}}{2}$$

Now we have two sequences $G_{n-1} = a\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ and

$$H_{n-1} = a\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$

In particular, $L_0 = 2$ but for G_0, H_0 to be two we need $a = 2$ .Again $G_n + H_n = G_{n-1} + H_{n-1} + G_{n-2} + H_{n-2}$ Therefore, $G_{n-1} + H_{n-1}$ also satisfies this recurrence.

$$G_{n-1}+H_{n-1}=a\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}+\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)$$

$L_0 = G_0 + H_0 = 2a$ for $n = 1$

$L_1 = G_1 + H_1 = a$ for $n = 2$

Which implies that $a = 1$.

Now $G_{n-1} + H_{n-1}$ satisfies the same recurrence as L_{n-1} and has the same initial values.

3. INVERSE TANGENT IDENTITIES FOR LUCAS NUMBERS

3.1 *MAIN RESULT*

Let $\tan^{-1}(x)$ denote the principal value of the inverse tangent function. The inverse tangent evaluations are based on a telescoping property of the inverse tangent function.

$$\tan^{-1}(x)-\tan^{-1}(y)=\tan^{-1}\left(\frac{x-y}{1+xy}\right),\ xy>-1 \quad \ldots(3.1)$$

Let $g(x)$ be a real function of one variable. Let $h(x)$ be of fixed sign and composite, $h(x)=h(g(x))$. Let $H(x)$ is given by the relation

$$H(x)=\frac{h(g(x))-h(g(x+1))}{1+h(g(x))h(g(x+1))} \quad \ldots(3.2)$$

$\therefore$ By telescoping we have

$$\sum_{n=1}^{\infty}\tan^{-1}H(n)=\sum_{n=1}^{\infty}\left(\tan^{-1}h(g(n))-\tan^{-1}\left(h(g(n+1))\right)\right)$$

Corollary 3.1.1: Let $h(x)=a/x$ and $g(n)=L_{mn}$, where $m=1,2,3,\ldots,\infty$

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{a(L_{mn+m}-L_{mn})}{a^2+L_{mn}L_{mn+m}}\right)=\tan^{-1}\left(\frac{a}{L_m}\right)$$

Corollary 3.1.2: Let $h(x)=ax+b$ and $g(n)=L_{mn}$, where $m=1,2,3,\ldots,\infty$

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{a(L_{mn+m}-L_{mn})}{a^2L_{mn}L_{mn+m}+ab(L_{mn+m}+L_{mn})+b^2+1}\right)=$$

$$\frac{\pi}{2}-\tan^{-1}\left(aL_m+b\right)$$

3.2 *PARTICULAR CASES AND SPECIAL VALUES*

Different combinations and values of the parameters *a* and *m* in the above corollary yield a variety of interesting particular cases. Here we are considering the two. Many other results can be obtained from it.

Results from Corollary 3.1.1:-

The case $a=1$, $m=1$ gives,

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{(L_{n+1}-L_n)}{(1+L_nL_{n+1})}\right)=\frac{\pi}{4}$$

$$\therefore\ \pi=4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{(L_{n+1}-L_n)}{(1+L_nL_{n+1})}\right) \quad \ldots(3.3)$$

The case $a=\sqrt{3}$, $m=1$ gives,

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{3}(L_{n+1}-L_n)}{(3+L_nL_{n+1})}\right)=\frac{\pi}{3}$$

$$\therefore\ \pi=3\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{3}(L_{n+1}-L_n)}{(3+L_nL_{n+1})}\right) \quad \ldots(3.4)$$

The case $a=\frac{1}{\sqrt{3}}$, $m=1$ gives,

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{3}(L_{n+1}-L_n)}{(1+3L_nL_{n+1})}\right)=\frac{\pi}{6}$$

$$\therefore\ \pi=6\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{3}(L_{n+1}-L_n)}{(1+3L_nL_{n+1})}\right) \quad \ldots(3.5)$$

The above relations (3.3), (3.4) and (3.5) represents π as the sum of infinite series involving Lucas numbers.

The case $a=\alpha$, $m=1$ gives the general identity

$$\therefore \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\alpha(L_{n+1}-L_n)}{(\alpha^2+L_nL_{n+1})}\right)=\tan^{-1}(\alpha) \quad \ldots(3.6)$$

Results from Corollary 3.1.2:-

The case $m=1$ gives,

$$\therefore \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{a(L_{n+1}-L_n)}{(a^2L_nL_{n+1}+ab(L_{n+1}+L_n)+b^2+1}\right)= \frac{\pi}{2}-\tan^{-1}(a+b) \quad \ldots(3.7)$$

From which the following evaluations are easily obtained.

The case $a=1$, $b=0$ gives,

$$\therefore \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{(L_{n+1}-L_n)}{(L_nL_{n+1}+1)}\right)=\frac{\pi}{4} \quad \ldots(3.8)$$

The above relation is equivalent to the relation equation (3.3).

The case $a=2$, $b=0$ gives,

$$\therefore \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2(L_{n+1}-L_n)}{(4L_nL_{n+1}+1)}\right)=\frac{\pi}{2}-\tan^{-1}(2) \quad \ldots(3.9)$$

The case $a=1$, $b=1$ gives,

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{(L_{n+1}-L_n)}{L_nL_{n+1}+L_{n+1}+L_n+2}\right)=\frac{\pi}{2}-\tan^{-1}(2) \quad \ldots(3.10)$$

The case $a=\alpha$, $b=0$ gives,

$$\therefore \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\alpha(L_{n+1}-L_n)}{(\alpha^2L_nL_{n+1}+1)}\right)=\frac{\pi}{2}-\tan^{-1}(\alpha) \quad \ldots(3.11)$$

Comparing with equation (3.6) we arrive at

$$\therefore \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\alpha(L_{n+1}-L_n)}{\alpha^2L_nL_{n+1}+1}\right)=\frac{\pi}{2}-\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\alpha(L_{n+1}-L_n)}{(\alpha^2+L_nL_{n+1})}\right) \quad \ldots(3.12)$$

By putting different values of constant a and b, different summation formulas can be obtained.

4. CONCLUSION

In this paper we have derived the Binet formula for Lucas numbers and a closed form evaluations of infinite series involving Lucas numbers are derived for finding the value of π.

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