Multiple W-E-K Fractional Integral of H- Function Pertaining to Srivastava Polynomials

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Abstract: **The aim of the present paper is to study and develop the multiple Wright-Erdelyi-Kober fractional integrals of H-function associated with Srivastava polynomials. Special cases, involving Multi-index Mittag-Leffer functions, Jacobi polynomials are considered. On account of the general nature of the functions involved, a large number of new integrals follow as special cases of the main finding.**
Keywords: **Multiple W-E-K fractional integral operator, H-function, Multi Index Mittag-Leffler function, Srivastava Polynomials.**

1. INTRODUCTION

Charles Fox [1] introduced function which is well-known in the literature as Fox's H-function or simply the H-function. This function has been defined and represented by means of the following Mellin-Barnes type contour integral [2]:

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right]$$

$$= H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1,\alpha_1),\dots,(a_p,\alpha_p) \\ (b_1,\beta_1),\dots,(b_q,\beta_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi\omega}\int_L \theta(\mathrm{s})\, z^{\mathrm{s}}\, d\mathrm{s} \qquad \dots(1.1)$$

where $\omega = \sqrt{-1},\ x \neq 0$ and

$$\theta(\mathrm{s}) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\mathrm{s})\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\mathrm{s})}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\mathrm{s})\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\mathrm{s})}. \qquad \dots(1.2)$$

Here m, n, p and q are non-negative integers satisfying $0 \le n \le p,\ 1 \le m \le q$. Also $\alpha_j\ (j=1,\dots,p)$ and $\beta_j\ (j=1,\dots,q)$ are assumed to be positive quantities for standardization purpose. Also $a_j\ (j=1,\dots,p)$ and $b_j\ (j=1,\dots,q)$ are complex numbers such that $\alpha_i(b_h+\nu) \neq \beta_h(a_i-\eta-1)$ for $\nu,\eta = 0,1,2,\dots; h=1,\dots,m; i=1,\dots,n$. L is contour separating the points $\mathrm{s} = \left(\frac{b_h+\nu}{\beta_h}\right)(h = 1,\dots,m\ ;\ \nu=0,1,2\dots)$ which are the poles of $\Gamma(b_h-\beta_h\mathrm{s})$ (h = 1,...,m), from the points $\mathrm{s} = \left(\frac{a_i-\eta-1}{\alpha_i}\right)(i = 1,\dots,n\ ;\ \eta=0,1,2,\dots)$ which are the poles of $\Gamma(1-a_i+\alpha_i\mathrm{s})$ (i = 1,...,n).

The Srivastava polynomials [3] is given by:

$$S_N^M[x] = \sum_{k=0}^{[N/M]} (-N)_{Mk}\, A_{N,k}\, \frac{x^k}{k!}, \qquad N = 0,1,2,\dots \qquad \dots(1.3)$$

where M is an arbitrary positive integer and the coefficients $A_{N,k}\ (N,k \ge 0)$ are arbitrary constants, real or complex. Here $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n=0 \\ \lambda(\lambda+1)\dots(\lambda+n-1), & \forall\, n \in \{i=1,\dots,r\}. \end{cases}$$

Let $m \ge 1$ be an integer; $\delta_i \ge 0, \gamma_i \in \Re, \beta_i > 0, i = 1,2,\dots,m$. We consider δ_i as multiorder of fractional integral; γ_i as multi-weight; β_i as additional parameter then multiple Wright-Erdelyi- Kober integral [4, 5] is defined by

$$\tilde{I}f(z)=I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}f(z)=\int_0^1 H_{m,m}^{m,0}\left[\sigma\left|\begin{matrix}(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_{1,m}\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_{1,m}\end{matrix}\right.\right]f(z\sigma)d\sigma;\ \text{if}\sum_{i=1}^{m}\delta_i>0$$

$$\tilde{I}f(z)=f(z)\quad\text{if}\ \ \delta_1=\delta_2=...=\delta_m=0 \qquad ...(1.4)$$

almost all the fractional calculus operators and most of their generalization fall in the generalized fractional calculus as special cases by taking multiplicies m = 1,2,… and special parameters.

The following lemma [4] will be required to establish our main result:

Lemma: For

$$\delta_i\geq 0,\gamma_i\in\Re,\beta_i>0,(i=1,2,...,m),\ p>\max_i[-\beta_i(\gamma_i+1)],\gamma_i\geq -1,\ p>0$$

$$I_{(\beta_i),m}^{(\gamma_i),(\delta_i)}[z^p]=\prod_{i=1}^{m}\frac{\Gamma(\gamma_i+1+\frac{p}{\beta_i})}{\Gamma(\gamma_i+\delta_i+1+\frac{p}{\beta_i})}z^p \qquad ...(1.5)$$

2. MAIN RESULT

$$I_{(\beta_i),l}^{(\gamma_i),(\delta_i)}\left[z^k(z+\xi)^{-\lambda}S_{N_1}^{M_1}\left(z^p(z+\xi)^{-q}\right)H_{P,Q}^{M,N}\left(z^\sigma(z+\xi)^{-\rho}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right)\right]$$

$$=z^k\xi^{-\lambda}\sum_{m=0}^{\infty}\sum_{k_1=0}^{[N_1/M_1]}\frac{(-1)^m}{m!}\frac{z^{m+pk_1}}{\xi^{m+qk_1}}(-N_1)_{M_1k_1}\frac{A(N_1,k_1)}{k_1!}$$

$$H_{P+1,Q+l+1}^{M,N+l+1}\left(\frac{z^\sigma}{\xi^\rho}\left|\begin{matrix}(1-\lambda-m-qk_1;\rho),(-\gamma_i-\frac{1}{\beta_i}(k+m+pk_1);\frac{\sigma}{\beta_i})_{1,l},(a_j,\alpha_j)_{1,P}\\(b_j,B_j)_{1,Q},(1-\lambda-qk_1;\rho),(-\gamma_i-\delta_i-\frac{1}{\beta_i}(k+m+pk_1);\frac{\sigma}{\beta_i})_{1,l}\end{matrix}\right.\right) \qquad ...(2.1)$$

Where

$$\delta_i\geq 0,\gamma_i\in\Re,\beta_i>0,\left(\gamma_i+\frac{1}{\beta_i}(k+m+pk_1)\right)\geq -1,(i=1,2,...,l).$$

$$\sigma,\rho\geq 0,l\geq 1\ \left|\arg z\right|<\frac{1}{2}A\pi\quad\text{and}\ A>0$$

where $A=\sum_{j=1}^{M}B_j+\sum_{j=1}^{N}a_j-\sum_{j=M+1}^{Q}B_j-\sum_{j=N+1}^{P}a_j$;

$$\text{Re}(k)+\sigma\min_{1\leq j\leq M}\left[\text{Re}\left(\frac{b_j}{B_j}\right)\right]>0$$

$$\text{Re}(\lambda)+\rho\min_{1\leq j\leq M}\left[\text{Re}\left(\frac{b_j}{B_j}\right)\right]>0$$

Proof:

To establish integral (2.1) we first express H-function in terms of Mellin-Barnes integral (1.1) and the Srivastava polynomials using (1.3). By interchanging the order of integration and summation and then using the lemma (1.5), we arrive at the desired result after little simplifications.

3. SPECIAL CASES

(i) On Setting

$M=N=P=1,Q=r+1;a_1=0,\alpha_1=1,b_1=0,\beta_1=1$;$b_i=1-\mu_i,B_i=1/\rho_i$;$i=2,..,r+1$ then

the main result takes the form

$$I_{(\beta_i),l}^{(\gamma_i),(\delta_i)}\left[z^k(z+\xi)^{-\lambda}S_{N_1}^{M_1}\left(z^p(z+\xi)^{-q}\right)E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}\left(-z^\sigma(z+\xi)^{-\rho}\right)\right]$$

$$=z^k\xi^{-\lambda}\sum_{m=0}^{\infty}\sum_{k_1=0}^{[N_1/M_1]}\frac{(-1)^m}{m!}\frac{z^{m+pk_1}}{\xi^{m+qk_1}}(-N_1)_{M_1k_1}\frac{A(N_1,k_1)}{k_1!}$$

$$H_{l+2,l+r+2}^{1,l+2}\left(\frac{z^\sigma}{\xi^\rho}\left|\begin{matrix}(1-\lambda-m-qk_1;\rho),(0,1),(-\gamma_i-\frac{1}{\beta_i}(k+m+pk_1);\frac{\sigma}{\beta_i})_{1,l}\\(0,1),(1-\mu_i,\frac{1}{\rho_i})_{1,r},(1-\lambda-qk_1;\rho),(-\gamma_i-\delta_i-\frac{1}{\beta_i}(k+m+pk_1);\frac{\sigma}{\beta_i})_{1,l}\end{matrix}\right.\right) \qquad ...(3.1)$$

which is valid under the conditions surrounding (2.1)

Where $E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)$ are the Multi-index Mittag-Leffler functions [6].

(ii)

Taking $M_1 = 1,\ A_{N_1,k_1} = \binom{N_1+\alpha'}{N_1} \dfrac{(\alpha'+\beta'+N_1+1)_{k_1}}{(\alpha'+1)_{k_1}}$,

the main result takes the form

$$I_{(\beta_i),l}^{(\gamma_i),(\delta_i)}\left[z^k (z+\xi)^{-\lambda} P_{N_1}^{(\alpha',\beta')}\left(1-2z^p(z+\xi)^{-q}\right)\right.$$

$$\left. .H_{P,Q}^{M,N}\left(z^\sigma (z+\xi)^{-\rho}\left| \begin{matrix} (a_j,\alpha_j)_{1,P} \\ (b_j,B_j)_{1,Q} \end{matrix}\right.\right)\right]$$

$$= z^k \xi^{-\lambda} \sum_{m=0}^{\infty} \sum_{k_1=0}^{N_1} \frac{(-1)^m}{m!k_1!} \frac{z^{m+pk_1}}{\xi^{m+qk_1}}$$

$$.(-N_1)_{k_1} \binom{N_1+\alpha'}{N_1} \frac{(\alpha'+\beta'+N_1+1)_{k_1}}{(\alpha'+1)_{k_1}}$$

$$.H_{P+1,Q+l+1}^{M,N+l+1}\left(\frac{z^\sigma}{\xi^\rho}\left| \begin{matrix} (1-\lambda-m-qk_1;\rho),(-\gamma_i-\frac{1}{\beta_i}(k+m+pk_1);\frac{\sigma}{\beta_i})_{1,l},(a_j,\alpha_j)_{1,P} \\ (b_j,B_j)_{1,Q},(1-\lambda-qk_1;\rho),(-\gamma_i-\delta_i-\frac{1}{\beta_i}(k+m+pk_1);\frac{\sigma}{\beta_i})_{1,l} \end{matrix}\right.\right)$$

valid under the conditions surrounding (2.1).

where $P_N^{(\alpha',\beta')}(y)$ is the Jacobi polynomials [7].

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4. CONCLUSION

The obtained result, besides being of very general character, have been put in a compact form and thus making it useful in applications. The present result provides interesting unification and extensions of a number of new results.

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