A Novel Video Encryption and Decryption Scheme Based on Discrete Wavelet Transform and Fractional Fourier Transform

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Abstract: **We propose three novel algorithms for video encryption and decryption. The algorithms also introduce additional one-level of encryption key into the existing methods. Data compression properties of the *DWT* (Discrete Wavelet Transform) are utilized to make the algorithm faster. The new algorithms retain the robustness of existing image encryption-decryption algorithms.**
A video is a collection of frames and a frame is similar to an image i.e. it can be segregated into three primary color channels *viz.* R, G and B. These channels are compressed by the proposed methods by using two times DWT_2 (2-D Discrete Wavelet Transform). The compressed frame-channels are encrypted using 2-D FRT (The 2-D fractional Fourier transform) and random phase masks in two successive iterations. The encrypted channels are merged by two times application of $IDWT_2$ (2-D Inverse Discrete Wavelet Transform), generating a color encrypted frame. Decryption process is the inverse of the encryption process.
Simulations are performed and the results of these simulations verify the proposals made in the new algorithms.
Period of study: - Wavelet Transform, Fractional Fourier Transform, Chaos, Random Phase Mask, Computational Complexity, video encryption.

1. INTRODUCTION

Many efficient image encryption-decryption methods have been proposed in recent past [1]. Video encryption-decryption algorithms are also of growing interest because of their importance in surveillance systems. In the past, a number of such methods have been proposed [2-5]. Efficient image encryption-decryption methods can also be utilized for video (frame) encryption-decryption. As the generalization of the conventional Fourier transform, the fractional Fourier transform has recently shown its potential in the field of optical security [1].

2. REVIEW OF FRT

The a^{th} order fractional Fourier transform is a linear operation defined by the integral

$$f_\alpha(u) \equiv \int_{-\infty}^{\infty} K_\alpha(u,u')f(u')du'$$

$$K_a(u,u') \equiv A_\alpha \exp\left[i\pi\left(\cot\alpha u^2 - 2\csc\alpha uu' + \cot\alpha u'^2\right)\right]$$

$$A_\alpha \equiv \sqrt{1-i\cot\alpha} \quad \alpha \equiv a\pi/2$$

When $a \neq 2j$ and $K_a(u, u') \Xi\ \delta(u-u')$ when $a = 4j$ and $K_a(u,u')\Xi\ \delta(u+u')$ when $a = 4j\pm2$, where j is an integer. The a^{th} order transform is sometimes referred to as the α^{th} order transform, a practice which will occasionally be found convenient when no confusion can arise [6].

3. WAVELET TRANSFORM (HAAR WAVELET)

The wavelet transform is a new mathematical tool developed mainly since the middle of the 1980s. It is efficient for local analysis of non stationary and fast transient wideband signals. The wavelet transform is a mapping of a time signal to the timescale joint representation, which is used in the short-time Fourier transform.

The wavelet transform can be easily extended to 2-D case for image processing applications. The wavelet transform of a 2-D frame $f(x, y)$ is a four-dimensional function.

$$W_f(s_x, s_y; u, v) = \frac{1}{\sqrt{s_x s_y}} \iint f(x,y)\psi\left(\frac{x-u}{s_x}; \frac{y-v}{s_y}\right)dxdy$$

It is reduced to a set of two-dimension functions of (u, v) with different scales, when the scale factors $s_x=s_y=s$. When $\psi(x, y)=\psi(r)$ with $r = (x^2+y^2)^{1/2}$, the wavelets are isotropic and have no selectivity for spatial orientation. Fig. 1 Represents the block diagram of two dimensional wavelet decomposition.

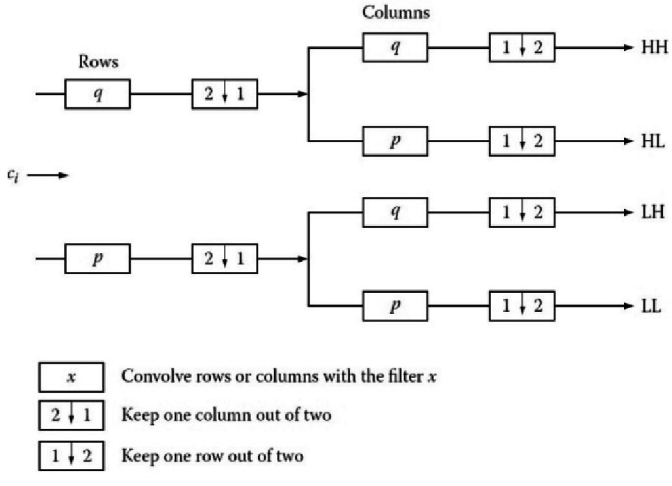

Fig 1 Schematic two-dimensional wavelet decomposition

Otherwise, the wavelet can have particular orientation. The wavelet can also be a combination of the 2-D wavelets with different particular orientations, so that the 2-D wavelet transform has orientation selectivity with quadrature mirror low-pass and high-pass filters p(n) and q(n).

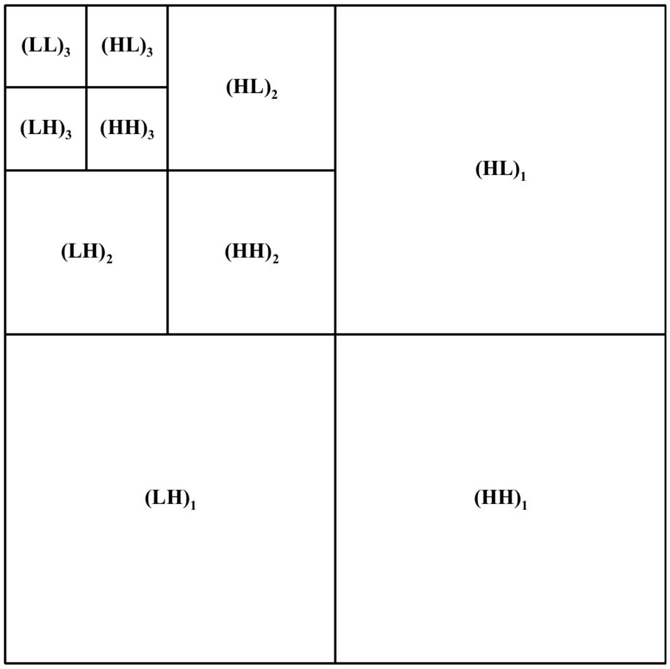

Fig 2 Presentation of the two-dimensional wavelet decomposition and high-pass filters $p(n)$ and $q(n)$

Then the pair of the 1-D filters are applied to each column of the two horizontally filtered images. The down-sampling by two is down sampling result in four sub-band images: (LL) for the low-pass filtered both horizontally and vertically image, (HH) for the highpass filtered both horizontally and vertically image, (LH) for lowpass filtered in horizontal direction and high-pass filtered in vertical direction image and (HL) for high-pass filtered in vertical direction and high-pass filtered in horizontal direction image, as shown in Fig. 2

4. PROPOSED FORMULAE

A video is a sequence $v^1v^2v^3v^4......v^m$ of m frames. Here we are considering that each frame $I = f^q(x,y)$ where $1 \le x, y \le n$ and $1 \le q \le m$ of size n x n consists of three primary color channels viz. RED, GREEN and BLUE *i.e.*

$$f^q(x,y) = \sum_{p=1}^{3} f_p^q(x,y)$$

$1 \le p \le 3$ and $1 \le q \le m$, where m = No. of frames in the video

For simplicity, we assume the frame size to be n x n. We propose two different frame encryption and decryption algorithms based on the Discrete Wavelet Transform, fractional Fourier transform and chaotic logistic-map and Kaplan-Yorke map. Each of these methods follows a similar sequence of operations but they differ in their implementation details. A general overview of the encryption and decryption process is as follows.

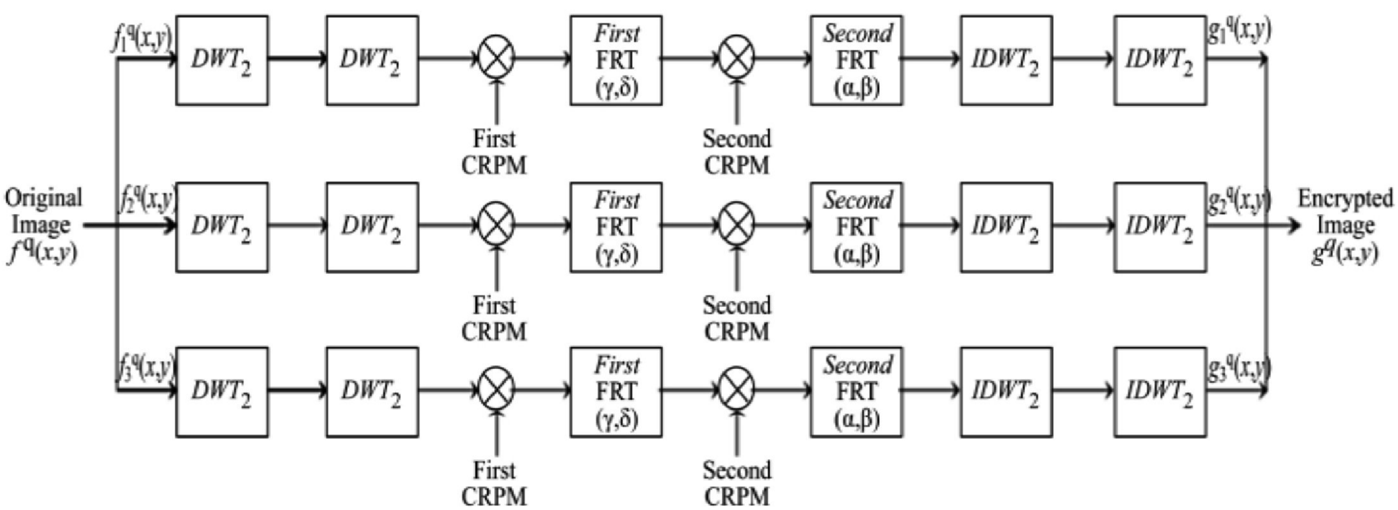

Fig 3 Encryption process using DWT_2 in proposed algorithm

As shown in fig 3, initially, the R, G and B channels of the input frame are segregated. Remaining operations are applied concurrently to these channels. Initially, the DWT_2 operation is performed twice over each channel to give us:

$$DWT_2\{DWT_2\{f_p^q(x,y)\}\}, \quad 1 \le p \le 3, 1 \le q \le m$$

This distribution is encoded by the first CRPM (Chaotic random phase mask) which is mathematically expressed as the phase function $\exp\left(i\frac{\pi}{2}S_1(x,y)\right)$, where $S_1(x,y)$ is a two-dimensional arrangement of random numbers generated by the chaos function. The first 2-*D* FRT operation is then performed over this to give us:

$$F_{\gamma,\delta}\left\{DWT_2\{DWT_2\{f_p^q(x,y)\}\} * \exp\left(i\frac{\pi}{2}S_1(x,y)\right)\right\}$$

where γ, δ are the fractional orders of the first 2-D FRT and * denote the element by element multiplication between two matrices of same order. The resultant is encoded by second CRPM which is mathematically expressed as the phase function $\exp\left(i\frac{\pi}{2}S_2(x,y)\right)$, where $S_2(x,y)$ is another two - dimensional arrangement of random numbers generated by the chaos function for a different seed value than the first CRPM. .

The second FRT operation is then performed over this to give us:

$$F_{\alpha,\beta}\left\{F_{\gamma,\delta}\left\{DWT_2\{DWT_2\{f_p^q(x,y)\}\} * \exp\left(i\frac{\pi}{2}S_1(x,y)\right)\right\} * \exp\left(i\frac{\pi}{2}S_2(x,y)\right)\right\}$$

where α, β are the fractional orders of the second 2-D FRT. Each channel is now operated two times with $IDWT_2$ to produce R, G and B channel of the encrypted frame. These channels are merged to produce the encrypted frame $g^q(x, y)$ as per the following formulae:

$$I_1 = g^q(x,y) = \sum_{p=1}^{3} g_p^q(x,y) = \sum_{p=1}^{3} IDWT_2\left\{IDWT_2\left\{F_{\alpha,\beta}\left\{F_{\gamma,\delta}\left\{DWT_2\{DWT_2\{f_p^q(x,y)\}\} * \exp\left(i\frac{\pi}{2}S_1(x,y)\right)\right\} * \exp\left(i\frac{\pi}{2}S_2(x,y)\right)\right\}\right\}\right\} \quad (1)$$

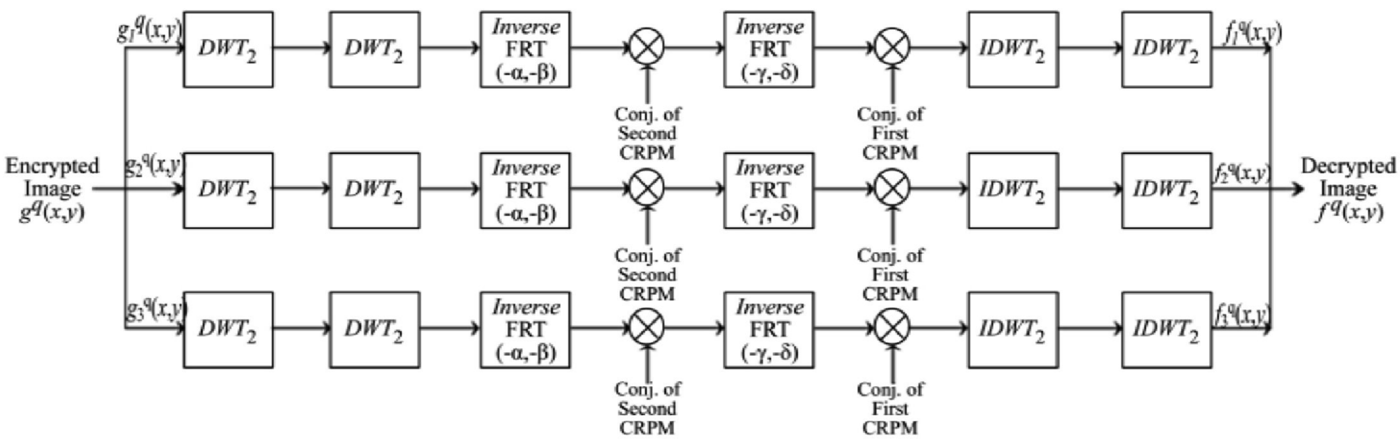

Fig 4 Decryption process using $IDWT_2$ in proposed algorithm

The decryption process, as shown in fig 4, is the inverse of the encryption process. The DWT_2 operation is performed twice over $g_P^q(x,y)$ the encrypted frame, to give us:

$$DWT_2\{DWT_2\{g_p^q(x,y)\}\}$$

The first inverse FRT (of order -α, -β) is now applied and followed by a multiplication by conjugate of second CRPM, thus giving us:

$$F_{-\alpha,-\beta}\{DWT_2\{DWT_2\{g(x,y)\}\}\} * conj\left(\exp\left(i\frac{\pi}{2}S_2(x,y)\right)\right)$$

The second FRT (of order -γ, -δ) is now performed and then conjugate of first CRPM is multiplied. The decrypted frame from this outcome is now obtained by performing IDWT2 as follows:

$$f'^q(x,y) = IDWT_2\left\{IDWT_2\left\{F_{-\gamma,-\delta}\left\{F_{-\alpha,-\beta}\{DWT_2\{DWT_2\{g_p^q(x,y)\}\}\} * conj\left(\exp\left(i\frac{\pi}{2}S_2(x,y)\right)\right)\right\} * conj\left(\exp\left(i\frac{\pi}{2}S_1(x,y)\right)\right)\right\}\right\}$$

(2)

5. PROPOSED ALGORITHMS

Algorithm 1: video encryption and decryption using DWT2 and FRT.

This algorithm uses DWT2 and Fα, β in encryption process and IDWT2 and F-α,-β in decryption process as per the general encryption and decryption schemes Both of the random phase functions are generated as a 2-D sequence of random numbers and they are not chaos based in this algorithm.

Computation complexity:

Let the input frame I be of size n x n. Analysis of the algorithm is divided into two phases viz. Encryption and Decryption.

Encryption: frame encryption process involves following steps:

1. Application of DWT2 twice on the primary color channels $f_p^q(x,y)$ of original frame
2. Encoding by CRPM.
3. Application of first 2-D FRT.
4. Encoding by second CRPM.
5. Application of second 2-D FRT
6. Application of IDWT2 twice on the R, G, and B channels obtained after step 5.

In step 1, each of the primary color channels of original frame is operated twice using DWT2, The asymptotic upper bound of this process is

$$O(n) + O\left(\frac{n}{2}\right) = O(n)$$

Thus step 1 takes $O(n)$ time and produces an output frame of size $\frac{n}{4}$ X $\frac{n}{4}$. Therefore, steps 2 to 6 are to be applied on a smaller sized frame than the original frame.

Computation of 2-*D FRT* of an image of size *n* x *n* is a $O(2n^3 + n\log_2 n)$ process. Also, the generation and multiplication of first and second random phase function is an $O(2n^2)$ process. But now, as the input frame size for steps 2 to 6 has reduced to $\frac{n}{4}$ X $\frac{n}{4}$,

(1) Step 2 takes $O\left(2\left(\frac{n}{4}\right)^2\right) = O\left(\frac{n^2}{8}\right)$ computation time.

(1) Step 3 takes $O\left(2\left(\frac{n}{4}\right)^3 + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) = O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right)$

Similar to step 2 and 3, step 4 and 5 also take $O\left(\frac{n^2}{8}\right)$ and $O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right)$ computation time, respectively. Step 6 involves computation of inverse wavelet transform, which is also a $O(N)$ function.

Therefore, the computation complexity of encryption is:

$$T_{encryption} = O(N) + O\left(\frac{n^2}{8}\right) + O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O\left(\frac{n^2}{8}\right)$$

$$+ O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O(n)$$

$$= O\left(\frac{n^3}{16} + \frac{n^2}{4} + \frac{n}{2}\log_2\left(\frac{n}{2}\right) + 2n\right)$$

$$\Rightarrow T_{encryption} = O\left(\frac{1}{2}\left(\frac{n}{2}\right)^3\right)$$

Decryption: frame decryption process involves following steps

1. Application of DWT_2 twice on the primary color components $g_p^q(x,y)$ of encrypted frame.
2. Application of 2-*D* inverse *FRT*.
3. Decoding by conjugate of second CRPM.
4. Application of 2-*D* inverse *FRT*.
5. Decoding by conjugate of first CRPM.
6. Application of $IDWT_2$ twice on the R, G, and B channels obtained after step 5.

In step 1, each of the primary color channels of encrypted frame is operated twice using DWT_2, The asymptotic upper bound of this process is $O(n) + O\left(\frac{n}{2}\right) = O(n)$ Thus step 1 takes $O(n)$ Time and produces an output encrypted frame of size $\frac{n}{4}$ X $\frac{n}{4}$

Therefore, steps 2 to 6 are to be applied on a smaller sized frame than the original encrypted frame.

(1) Step 2 takes $O\left(2\left(\frac{n}{4}\right)^3 + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right)$

$= O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right)$ time.

(2) Step 3 takes $O\left(2\left(\frac{n}{4}\right)^2\right) = O\left(\frac{n^2}{8}\right)$ time.

Similar to step 2 and 3, step 4 and 5 also take $O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right)$ and $O\left(\frac{n^2}{8}\right)$ computation time, respectively. Step 6 involves computation of inverse wavelet transform, which is an $O(n)$ process. Therefore, the computation complexity of decryption is:

$$T_{decryption} = O(n) + O\left(\frac{n^2}{8}\right) + O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O\left(\frac{n^2}{8}\right)$$

$$+ O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O(n)$$

$$= O\left(\frac{n^3}{16} + \frac{n^2}{4} + \frac{n}{2}\log_2\left(\frac{n}{4}\right) + 2n\right)$$

$$\Rightarrow T_{decryption} = O\left(\frac{1}{2}\left(\frac{n}{2}\right)^3\right)$$

Thus, the computation complexity of algorithm 1 is evaluated to

$$T_1 = T_{encryption} + T_{decryption} = O\left(\frac{n}{2}\right)^3$$

There are m frames is a video therefore the total computation time for encryption & decryption process is

$$T_1 = O\left(m\frac{n^3}{8}\right)$$

$$\Rightarrow T_1 \propto m, T_1 \propto n^3$$

Algorithm 2: ***video encryption and decryption using DWT2 and FRT with logistic map.***

This algorithm uses DWT_2, $F_{\alpha,\beta}$ in encryption process and $IDWT_2$, $F_{-\alpha,-\beta}$ in decryption process as per the general encryption and decryption schemes shown in Fig. 1 and Fig. 2.

Computational complexity

The analysis of this algorithm is also performed in two phases as in case of the Algorithm 1:

Encryption: Step 1, 3, 5 and 6 of encryption are similar to that of algorithm 1 and so are their computation times. Step 2 is Encoding by first random phase function and it takes

$O\left(2\left(\frac{n}{4}\right)^2\right) = O\left(\frac{n^2}{8}\right)$ time. Similarly step 4 is Encoding by second random phase function and it takes $O\left(\frac{n^2}{8}\right)$ time.

Therefore, the computation complexity of entire encryption phase is computed as follows:

$$T_{Encryption} = O(n) + O\left(\frac{n^2}{8}\right) + O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O\left(\frac{n^2}{8}\right)$$

$$+ O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O(n)$$

$$= O\left(\frac{n^3}{16} + \frac{n^2}{4} + \frac{n}{2}\log_2\left(\frac{n}{4}\right) + 2n\right)$$

$$\Rightarrow T_{Encryption} = O\left(\frac{1}{2}\left(\frac{n}{2}\right)^3\right)$$

Decryption: Steps 1, 2, 4, 6 and their computation times of algorithm 2 are similar to that of the Algorithm 1. Step 3 and 5 perform decoding by conjugate of second and first CRPM respectively and each of them takes $O\left(\frac{n^2}{8}\right)$ time.

Therefore

$$T_{deryption} = O(n) + O\left(\frac{n^2}{8}\right) + O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O\left(\frac{n}{8}\right)$$

$$+ O\left(\frac{n^3}{32} + \frac{n}{4}\log_2\left(\frac{n}{4}\right)\right) + O(n)$$

$$= O\left(\frac{n^3}{16} + \frac{n^2}{8} + \frac{n}{2}\log_2\left(\frac{n}{4}\right) + 2n\right)$$

$$\Rightarrow T_{Decryption} = O\left(\frac{1}{2}\left(\frac{n}{2}\right)^3\right)$$

Thus, the computational complexity of algorithm 2 is evaluated to:

$$T_2 = T_{encryption} + T_{decryption} = O(n^3)$$

There are m frames is a video therefore the total computation time for encryption & decryption process is

$$T_2 = O\left(m\frac{n^3}{8}\right)$$

$$\Rightarrow T_2 \propto m, T_2 \propto n^3$$

Algorithm 3 - *video encryption and decryption using DWT2 and FRT with Kaplan-Yorke map.*

This algorithm is similar to the Algorithm 2; the only difference lies in the chaotic function used to generate the random phase mask which is two-dimensional Chaotic Kaplan-Yorke map

The expression for the computational complexity of algorithm 3 is evaluated to:

$$T_3 = T_{encryption} + T_{decryption} = O\left(\frac{n^3}{8}\right)$$

$$T_3 = O\left(m\frac{n^3}{8}\right)$$

$$\Rightarrow T_3 \propto m, T_3 \propto n^3$$

6. SIMULATION RESULTS

In order to investigate the quality of encryption, decryption and efficiency of proposed algorithms, digital simulations were performed using matlab 2008 b.

The input video chosen for analysis is nasa_117_256.avi (AVI Video; Size = 283 KB; pixel by pixel resolution = 256X256; no. of frames = 117). The video, nasa_117_256.avi was encrypted using algorithm 1, 2 and 3 for $\alpha = \beta = \gamma = \delta = 0.5$ fractional orders of *FRT*. The frames encrypted using algorithm 2 and 3 are shown in Fig 6(a) and 6(b). These frames can be decrypted on any fractional order of *FRT*, but the restored frame may differ from the input frame depending on the order of *FRT*. The decrypted frames obtained using incorrect fractional orders of inverse *FRT* are shown in Fig 7(a) and 7(b). Fig. 8(a) and 8(b) show decrypted images obtained when correct fractional orders of inverse FRT are used.

Fig 5 Input frame to algorithm 1, 2 and 3 (nasa_117_256.avi, 256X256, color).

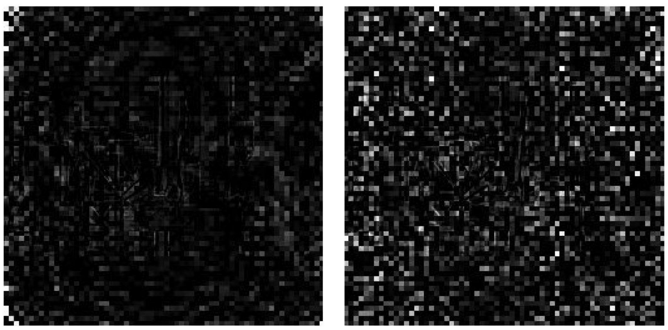

Fig 6 Encrypted frames of size 256X256 by (a) Algorithm 2 and (b) Algorithm 3. The fractional orders of *FRT* are α = β = γ = δ = 0.5

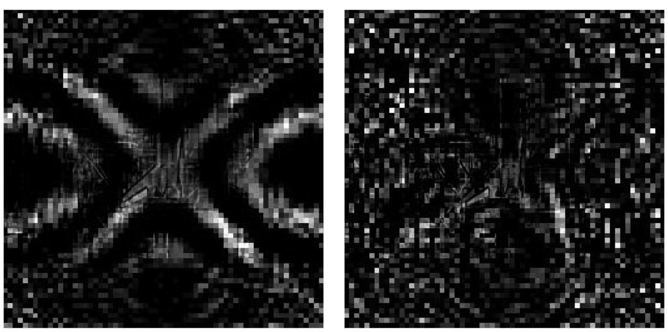

Fig 7 Decrypted frames of size 256X256 decrypted on an incorrect fractional order of inverse *FRT* by (a) Algorithm 2 and (b) Algorithm 3. The fractional orders of *FRT* are α = β = γ = δ = 0.4

Fig 8 Decrypted frames of size 256X256 decrypted on appropriate fractional order of inverse *FRT* by (a) Algorithm 2 and (b) Algorithm 3. The fractional orders of *FRT* are α = β = γ = δ = 0.5

TABLE 1: Computation times of Algo 1, Algo 2 and Algo 3 with no. of frames

No. of frames (m)	Computation Time (in Seconds)		
	Algo 1	Algo 2	Algo 3
1	2.71	2.8	2.79
3	7.39	7.62	7.62
7	17.08	17.45	17.34
14	33.85	34.72	34.69
29	75.73	73.25	72.88
58	148.66	152.51	154.28

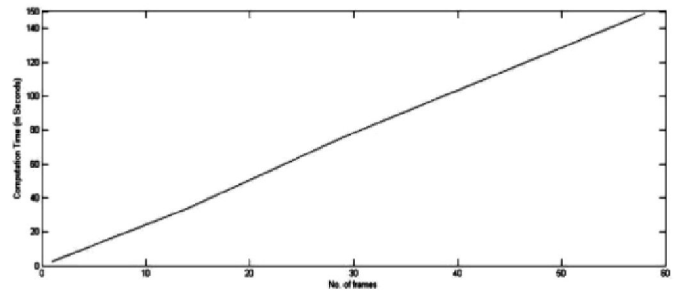

Fig 9 Graph showing computation time of Algorithm 1.

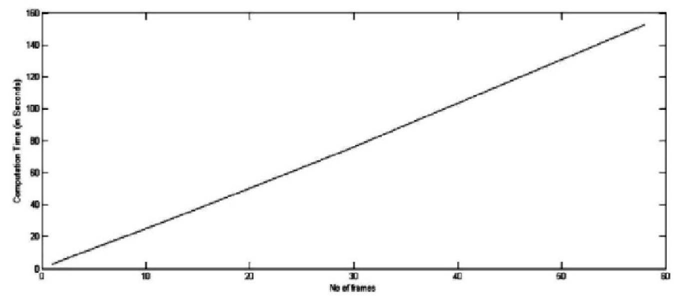

Fig 10 Graph showing computation time of Algorithm 2.

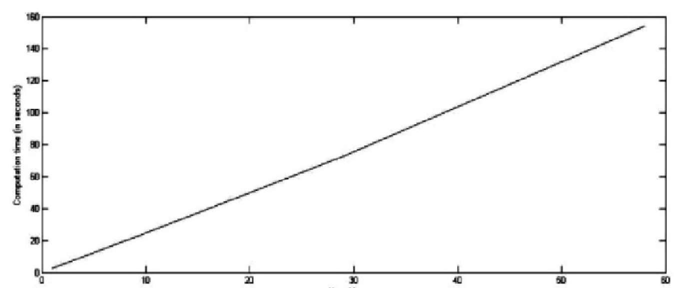

Fig 11 Graph showing computation time of Algorithm 3.

7. CONCLUSION

We have proposed three algorithms for video encryption and decryption and their behavior has been analyzed, based on the computation time required by the algorithms. Each of the algorithm works on a strategy that the data size (or frame size) for encryption and decryption is reduced by a factor of 16 (n x n to $\frac{n}{4}$X$\frac{n}{4}$) than the existing algorithms. For this purpose we have efficiently utilized the data compression characteristics of the discrete wavelet transform.

On the basis of graphs shown in fig 9, 10 and 11 following claims are justified.

$T_1 \propto m$, $T_2 \propto m$ and $T_3 \propto m$

The proposed algorithms retain the robustness of the original image encryption-decryption algorithms. By this we mean that once we encrypt a video frame for a particular fractional order of *FRT*, decryption of this encrypted frame is only possible, when the selected fractional order for decryption is exactly the suitable for decryption. Results are verified for fractional order of 2-D *FRT* = 0.5 and fractional order of 2-D *IFRT* = 0.4, 0.5.

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