Fractional Calculus Operator Associated With Wright's Function

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Abstract: **The present paper aims at the study and derivation of Saigo generalized fractional integral operator involving product of Multivariable H-function generalized polynomials and Wright function. On account of the most general nature of the operator, H-function ,generalized polynomial and Wright's function occurring in the main result, a large number of known and new results involving Rieman-Liouville, Erdélyi- Kober Fractional differential operators,Bessel function, Mittag-leffler function follows as special cases of our main finding.**

Key Words: **Saigo fractional integral operator, Wright's hypergeometric function, H-function of several complex variables, generalized polynomial, Appel function.**

1. INTRODUCTION

The H-function of several complex variables introduced by Srivastava and Panda [1] is defined and represented in the following manner:

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}]\\ [(c):\psi';\ldots;\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}]\end{matrix}\ z_1,\ldots,z_r\right]$$

$$=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}T(s_1,\ldots,s_r)\,R_1(s_1)\cdots R_r(s_r)\,z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \quad \ldots(1.1)$$

where

$$w=\sqrt{(-1)},\qquad R_i(s_i)=\frac{\prod_{j=1}^{u^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{v^{(i)}}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}s_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}s_i)},$$

$\forall\ (i=1,2,\ldots,r)$,

$$T(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{\lambda}\Gamma(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i)}{\prod_{j=\lambda+1}^{A}\Gamma(a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i)\prod_{j=1}^{C}\Gamma(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}s_i)},$$

and an empty product is interpreted as unity.

Wright's hypergeometric function [2] is defined by

$$_p\psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j m)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j m)}\frac{z^m}{m!} \quad \ldots(1.2)$$

where

$1+\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j\geq 0$, $\alpha_j\,(j=1,\ldots,p)$ and $\beta_j\,(j=1,\ldots,q)$ are positive real numbers.

The generalized polynomial due to Srivastava [3] is defined as

$$S^{M_1,\ldots,M_l}_{N_1,\ldots,N_l}[x_1,\ldots,x_l]=\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_l=0}^{[N_l/M_l]}(-N_1)_{M_1k_1}\cdots(-N_l)_{M_lk_l}\,.B(N_1,k_1;\ldots;N_l,k_l)\frac{(x_1)^{k_1}}{k_1!}\cdots\frac{(x_l)^{k_l}}{k_l!} \quad \ldots(1.3)$$

where $N_{i'}=0,1,2,\ldots\ (\forall\ i'=1,\ldots,l)$, $M_1,\ldots,M_l$ are arbitrary positive integers and the coefficients $B(N_1,k_1;\ldots;N_l,k_l)$ are arbitrary constants, real or complex

The Saigo fractional integral operator [4, 5] is defined as as

$$I_{0,x}^{p,q,\gamma}f(x)=\begin{cases}\dfrac{x^{-p-q}}{\Gamma(p)}\displaystyle\int_0^x(x-t)^{p-1}F(p+q,-\gamma;p;1-\frac{t}{x})f(t)dt & (\mathrm{Re}(p)>0)\\[2ex] \dfrac{d^r}{dx^r}I_{0,x}^{p+r,q-r,\gamma-r}f(x), & (\mathrm{Re}(p)\leq 0,\ 0<\mathrm{Re}(p)+r\leq 1, r=1,2,\ldots)\end{cases} \quad \ldots(1.4)$$

where F is the Gauss hypergeometric function.

Saigo fractional integral operator contains as special cases, the Riemann-Liouville and Erdély- Kober operators of Fractional Integration of order $\alpha>0$ [6, 7].

$$I_{0,z}^{\alpha,-\alpha,-\alpha}f(z)=R^{\alpha}f(z)=\frac{z^{\alpha}}{\Gamma(\alpha)}\int_0^1(1-t)^{\alpha-1}f(tz)dt$$

$$z^{-\alpha-\gamma}I_{0,z}^{\alpha,-\alpha-\gamma,-\alpha}f(z)=I_1^{\gamma,\alpha}f(z)=\frac{1}{\Gamma(\alpha)}\int_0^1(1-t)^{\alpha-1}t^{\gamma}f(zt)dt \quad (\alpha>0,\gamma\in\mathbf{R})$$

Let $\alpha, \alpha', \beta, \beta'\in$ R and $\gamma>0$, then Saigo generalized fractional integral operator [4] of a function $f(x)$ is defined by

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}f(z)=\frac{z^{-\alpha}}{\Gamma(\gamma)}\int_0^z(z-t)^{\gamma-1}t^{-\alpha'}\,.F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{z},1-\frac{z}{t})f(t)dt \quad \gamma>0 \quad \ldots(1.5)$$

Where *f*(*z*) is analytic in a simply connected region of z-plane. Principal value for $0 \le \arg(z-t) \le 2\pi$ is denoted by $(z-t)^{\gamma-1}$

The Appell hypergeometric function of third type, also known as Horn's $\mathbf{F_3}$ function is defined as

$$F_3(\alpha,\alpha';\beta,\beta';\gamma;z,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{z^m t^n}{m!n!} \qquad |z|<1,|t|<1$$

Following Lemma [4]; see also [8] will be required in the sequel:

Lemma: Let

$\operatorname{Re}(\gamma)>0, k>\max[0,\operatorname{Re}(\alpha+\alpha'+\beta-\gamma),\operatorname{Re}(\alpha'-\beta')]-1$

then

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}[z^k]=\frac{\Gamma(1+k)\Gamma(1+k-\alpha'+\beta')\Gamma(1+k-\alpha-\alpha'-\beta+\gamma)}{\Gamma(1+k+\beta')\Gamma(1+k-\alpha'-\beta+\gamma)\Gamma(1+k-\alpha-\alpha'+\gamma)}z^{k-\alpha-\alpha'+\gamma} \qquad ...(1.6)$$

2. RESULT

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}(p-qt)^{\rho}S_{N_1,\ldots,N_l}^{M_1,\ldots,M_l}\begin{pmatrix}t^{u_1}(p-qt)^{-v_1}\\ \vdots\\ t^{u_l}(p-qt)^{-v_l}\end{pmatrix}{}_m\Psi_n\left(t^{\lambda_1}(p-qt)^{-\mu}\right)\right.$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}\left[\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:[b':\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]\\ [(c):\psi';\ldots;\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}]\end{matrix}\right.$$

$$\left.\begin{matrix}z_1t^{\delta_1}(p-qt)^{-\eta_1}\\ \vdots\\ z_rt^{\delta_r}(p-qt)^{-\eta_r}\end{matrix}\right]$$

$$=\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_l=0}^{[N_l/M_l]}(-N_1)_{M_1k_1}\cdots(-N_l)_{M_lk_l}B(N_1,k_1;\ldots;N_l,k_l)\frac{(x_1)^{k_1}}{k_1!}\cdots\frac{(x_l)^{k_l}}{k_l!}$$

$$.p^{\rho-\sum_{i=1}^{l}v_ik_i}\,t^{\sigma+\sum_{i=1}^{l}u_ik_i-\alpha-\alpha'+\gamma}\;.H_{A+4,C+4:[B',D'];\ldots;[B^{(r)},D^{(r)}];(m,n+1);(0,1)}^{0,\lambda+4:(u',v');\ldots;(u^{(r)},v^{(r)});(1,m);(1,0)}$$

$$\left[\begin{matrix}z_1t^{\eta_1}\\ \vdots\\ z_rt^{\eta_r}\\ -t^{\lambda}\\ -t\end{matrix}\right|\begin{matrix}(1+\rho-\sum_{i=1}^{l}v_ik_i;q_1,q_2,\ldots,q_r,\mu,1),\\ (1+\rho-\sum_{i=1}^{l}v_ik_i;q_1,q_2,\ldots,q_r,\mu,0),\end{matrix}$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha'-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha'+\beta'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha+\alpha'+\beta-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),[(a):\theta';\ldots;\theta^{(r)},0,\ldots,0]:$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha+\alpha'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),[(c):\psi';\ldots;\psi^{(r)},0,\ldots,0]:$$

$$\left.\begin{matrix}[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}];[1-a_i:\alpha_i];\ \_\ ;\ \_\\ [d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}];[1-b_j:\beta_j];(0,1),(0,1)\end{matrix}\right] \qquad \ldots(2.1)$$

Provided

1) $\alpha,\alpha',\beta,\beta',\gamma,\mu,\lambda_1,\sigma,\rho\in C;u_1,u_2,\ldots,u_l,v_1,v_2,\ldots,v_l>0$

2) $\gamma>0,\operatorname{Re}(\sigma)+\sum_{i'=1}^{r}\delta_i\min_{1\le j\le u^{(r)}}\left[\operatorname{Re}(\frac{d_j^{(i')}}{\delta_j^{(i')}}\right]+1>\max\{0,\alpha'-\beta',\alpha+\beta-\gamma\}-1$

3) $\left|\frac{q}{p}t\right|<1$

3. PROOF

In order to prove (2.1), we first express the generalized polynomials in series form (1.3) , the H-function in terms of Mellin-Barnes type of contour integrals (1.1), wright function and then interchange the order of summations, integration and fractional integral operator, which is permissible under the stated conditions. Now using the result (1.6) we arrive at the desired result after a little simplification.

4. INTERESTING SPECIAL CASES

On account of the most general character of the H-function, generalized polynomials and wright's function occurring in the main result, many special cases of the result can be derived but, for the sake of brevity, a few interesting special cases are recorded here.

(i) Setting $m=1=n,a_1=1=\alpha_1,b_1=B,\beta_1=A'$ in (2.1), we get

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}(p-qt)^{\rho}S_{N_1,\ldots,N_l}^{M_1,\ldots,M_l}\begin{pmatrix}t^{u_1}(p-qt)^{-v_1}\\ \vdots\\ t^{u_l}(p-qt)^{-v_l}\end{pmatrix}E_{A',B}\left(t^{\lambda_1}(p-qt)^{-\mu_1}\right)\right.$$

$$\left.H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}]\\ [(c):\psi';\ldots;\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}]\end{matrix}\middle|\begin{matrix}z_1t^{\delta_1}(p-qt)^{-\eta_1}\\ \vdots\\ z_rt^{\delta_r}(p-qt)^{-\eta_r}\end{matrix}\right)\right]$$

$$=\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_l=0}^{[N_l/M_l]}(-N_1)_{M_1k_1}\cdots(-N_l)_{M_lk_l}B(N_1,k_1;\ldots;N_l,k_l)\frac{(x_1)^{k_1}}{k_1!}\cdots\frac{(x_l)^{k_l}}{k_l!}$$

$$p^{\rho-\sum_{i=1}^{l}v_ik_i}\,t^{\sigma+\sum_{i=1}^{l}u_ik_i-\alpha-\alpha'+\gamma}\;H_{A+4,C+4:[B',D'];\ldots;[B^{(r)},D^{(r)}];(1,2);(0,1)}^{0,\lambda+4:(u',v');\ldots;(u^{(r)},v^{(r)});(1,1);(1,0)}$$

$$\left[\begin{matrix}z_1t^{\eta_1}\\ \vdots\\ z_rt^{\eta_r}\\ -t^{\lambda}\\ -t\end{matrix}\right|\begin{matrix}(1+\rho-\sum_{i=1}^{l}v_ik_i;q_1,q_2,\ldots,q_r,\mu,1),\\ (1+\rho-\sum_{i=1}^{l}v_ik_i;q_1,q_2,\ldots,q_r,\mu,0),\end{matrix}$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i;\eta_1,\eta_2,\ldots,\eta_r,\lambda,1),(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha'-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda,1),$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda,1),(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha'+\beta'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda,1),$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha+\alpha'+\beta-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda,1),[(a):\theta';\ldots;\theta^{(r)},0,\ldots,0]:$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha+\alpha'-\gamma;\eta_1,\eta_2,...,\eta_r,\lambda,1),[(c):\psi';...;\psi^{(r)},0,...,0]:$$

$$\left.\begin{matrix}[b':\phi'];...;[b^{(r)}:\phi^{(r)}];[0:1];\ \_\ ;\ \_\\ [d':\delta'];...;[d^{(r)}:\delta^{(r)}];[1-B:A'];(0,1),(0,1)\end{matrix}\right] \quad ...(3.1)$$

valid under the conditions derived from those mentioned for (2.1).

(ii) On taking $\lambda = A = C = 0$, eq. (2.1) reduces to the following form

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}(p-qt)^{\rho}S_{N_1,...,N_l}^{M_1,...,M_l}\begin{pmatrix}t^{u_1}(p-qt)^{-v_1}\\ \vdots\\ t^{u_l}(p-qt)^{-v_l}\end{pmatrix}{}_m\Psi_n\left(t^{\lambda_1}(p-qt)^{-\mu_1}\right)\right.$$

$$\left.\cdot\prod_{i=1}^{r}H_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}}\left(z_it^{\delta_i}(p-qt)^{-\eta_i}\left|\begin{matrix}[(b^{(i)}):\phi^{(i)}]\\ [(d^{(i)}):\delta^{(i)}]\end{matrix}\right.\right)\right]$$

$$=\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_l=0}^{[N_l/M_l]}(-N_1)_{M_1k_1}\cdots(-N_l)_{M_lk_l}B(N_1,k_1;...;N_l,k_l)\frac{(x_1)^{k_1}}{k_1!}\cdots\frac{(x_l)^{k_l}}{k_l!}$$

$$p^{\rho-\sum_{i=1}^{l}v_ik_i}\ t^{\sigma+\sum_{i=1}^{l}u_ik_i-\alpha-\alpha'+\gamma}\ H_{4,4:[B',D'];...;[B^{(r)},D^{(r)}];(m,n+1);(0,1)}^{0,4:(u',v');...;(u^{(r)},v^{(r)});(1,m);(1,0)}$$

$$\left[\begin{matrix}z_1t^{\eta_1}\\ \vdots\\ z_rt^{\eta_r}\\ -t^{\lambda}\\ -t\end{matrix}\right|\begin{matrix}(1+\rho-\sum_{i=1}^{l}v_ik_i;q_1,q_2,...,q_r,\mu,1),\\ (1+\rho-\sum_{i=1}^{l}v_ik_i;q_1,q_2,...,q_r,\mu,0),\end{matrix}$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i;\eta_1,\eta_2,...,\eta_r,\lambda,1),(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha'-\beta';\eta_1,\eta_2,...,\eta_r,\lambda,1),$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i-\beta';\eta_1,\eta_2,...,\eta_r,\lambda,1),(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha'+\beta'-\gamma;\eta_1,\eta_2,...,\eta_r,\lambda,1),$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha+\alpha'+\beta-\gamma;\eta_1,\eta_2,...,\eta_r,\lambda,1),-:$$

$$(-\sigma-\sum_{i=1}^{l}u_ik_i+\alpha+\alpha'-\gamma;\eta_1,\eta_2,...,\eta_r,\lambda,1),-:$$

$$\left.\begin{matrix}[b':\phi'];...;[b^{(r)}:\phi^{(r)}];[1-a_i:\alpha_i];\ \_\ ;\ \_\\ [d':\delta'];...;[d^{(r)}:\delta^{(r)}];[1-b_j:\beta_j];(0,1),(0,1)\end{matrix}\right] \quad ...(3.2)$$

valid under the conditions derived from those mentioned for (2.1).

5. CONCLUSION

The results derived in the present paper are quite general in nature so a large number of known and new results involving Riemann-Liouville, Erdélyi- Kober Fractional differential operators,Bessel function, Mittag-leffler function etc. can be obtained from it.

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