Double Inequalities for the I-function

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Abstract: **In the present paper, we establish some double inequalities for I-function of one variable with the help of Nguyen Van Vinh and Ngo Phuoc Nguyen Ngoc inequality.**

1. INTRODUCTION

Rathie [1] has introduced the I-function which is a new generalization of the $\overline{H}$-function which was the generalization of the familiar H-function of Fox [2]. The I-function contains the exact partition function of the Gaussian model in statistical mechanics, functions useful in testing hypothesis and several others as its special cases. The I-function will be defined and represented by the following Mellin-Barnes type contour integral .

$$I_{P,Q}^{M,N}[z] = I_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j, A_j;\alpha_j)_{1,N}, (a_j, A_j;\alpha_j)_{N+1,P} \\ (b_j, B_j;\beta_j)_{1,M}, (b_j, B_j;\beta_j)_{M+1,Q} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i}\int_L \theta(s) z^s ds, \tag{1.1}$$

where θ (s) is given by :

$$\theta(s) = \frac{\prod_{j=1}^{M} \Gamma^{\beta_j}(b_j - B_j s) \prod_{j=1}^{N} \Gamma^{\alpha_j}(1 - a_j + A_j s)}{\prod_{j=M+1}^{Q} \Gamma^{\beta_j}(1 - b_j + B_j s) \prod_{j=N+1}^{P} \Gamma^{\alpha_j}(a_j - A_j s)} \tag{1.2}$$

Also,

(i) $z \neq 0$

(ii) $i = \sqrt{-1}$

(iii) M,N,P,Q are integers satisfying \
$0 \le M \le Q, 0 \le N \le P.$

(iv) L is a suitable contour in the complex plane.

(v) An empty product is to be interpreted as unity.

(vi) A_j , j = 1,2,…,P; B_j , j = 1,2,…,Q ; α_j , j = 1,2,…,P and β_j , j = 1,2,…,Q are real positive numbers.

(vii) a_j , j = 1,2,…,P and b_j , j = 1,2,…,Q are complex numbers.

There are three different contours L of integration

(a) L goes from $\sigma - i\infty$ to $\sigma + i\infty$, (σ is real) so that all the singularities of $\Gamma^{\beta_j}(b_j - B_j s)$, j = 1,2,…,M lie to the right, and all the singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s)$,
j = 1,2,…,N lie to the left of L.

(b) L is a loop beginning and ending at $+\infty$ and encircling all the singularities of $\Gamma^{\beta_j}(b_j - B_j s)$, j = 1,2,…,M once in the clockwise direction, but none of the singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s)$, j =1,2,…,N.

(c) L is a loop beginning and ending at $-\infty$ and encircling all the singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s)$, j = 1,2,…,N once in the anti-clockwise direction, but none of the singularities of $\Gamma^{\beta_j}(b_j - B_j s)$, j = 1,2,…,M

In short (1.1) will be denoted by,

$$I_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j, A_j;\alpha_j)_{1,P} \\ (b_j, B_j;\beta_j)_{1,Q} \end{matrix} \right. \right]$$

For α (and / or β_j) not an integer, the poles of the gamma functions of the numerator in (1.2) are converted to branch points. The branch cuts can be chosen so that the path of integration c an be distorted for each of the three contours L mentioned above as long as there is no coincidence of poles from any $\Gamma(b_j - B_j s)$, and $\Gamma(1 - a_j + A_j s)$ pair.

The sufficient conditions for convergence of (1.1) are following :

$$\theta = \sum_{j=1}^{M} |B_j\beta_j| + \sum_{j=1}^{N} |A_j\alpha_j| - \sum_{j=M+1}^{Q} |B_j\beta_j| - \sum_{j=N+1}^{P} |A_j\alpha_j| > 0 \tag{1.3}$$

and $$|\arg z| < \theta\frac{\pi}{2} \tag{1.4}$$

where θ is given by (1.3).

Evidently when the exponents $\alpha_j = 1\,(j = N+1,\dots,P)$ and $\beta_j = 1\,(j=1,\dots,M)$ the I-function reduces to the $\bar{H}$-function given by Inayat Hussain and when the exponents $\alpha_j = 1\,(j=1,\dots,P)$ and $\beta_j = 1\,(j=1,\dots,Q)$ the I-function reduces to the well known Fox's H - function as follows :

$$I_{P,Q}^{M,N}\left[Z\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,N},(a_j,A_j;1)_{N+1,P}\\(b_j,B_j;1)_{1,M},(b_j,B_j;\beta_j)_{M+1,Q}\end{matrix}\right.\right] \tag{1.5}$$

$$= \bar{H}_{P,q}^{M,N}\left[Z\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,N},(a_j,A_j)_{N+1,P}\\(b_j,B_j)_{1,M},(b_j,B_j;\beta_j)_{M+1,Q}\end{matrix}\right.\right] \tag{1.6}$$

and

$$I_{P,q}^{M,N}\left[Z\left|\begin{matrix}(a_j,A_j;1)_{1,P}\\(b_j,B_j;1)_{1,Q}\end{matrix}\right.\right] = H_{P,q}^{M,N}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]$$

2. DOUBLE INEQUALITY FOR THE I-FUNCTION

(1) 1st\ Inequality:-

$$I_{p+n,q+1}^{d,u+n}\left[z\left|\begin{matrix}(\alpha_i,u_i)_{i=i,n},\{a_p,A_p\}\\\{b_q,B_q\},\left(-\beta-\sum_{i=1}^{n}a_i;u+\sum_{i=1}^{n}(u_i)\right)\end{matrix}\right.\right]$$

$$\le$$

$$I_{p+n,q+1}^{d,u+n}\left[z\left|\begin{matrix}(\alpha_i,u_i)_{i=i,n},\{a_p,A_p\}\\\{b_q,B_q\},\left(-\beta-\sum_{i=1}^{n}a_i;u+\sum_{i=1}^{n}(u_i)\right)\end{matrix}\right.\right] \backslash$$

$$\le I_{p,q+1}^{d,u}\left[z\left|\begin{matrix}\{a_p,A_p\}\\\{b_q,B_q\},(1-\beta,u)\end{matrix}\right.\right] \tag{2.1}$$

Provided that;

$x \in [0,1),$

$$\operatorname{Re}\left[\beta + \min_{1\le j\le d} u\left(\frac{b_j}{B_j}\right)\right] > 1$$

$$\operatorname{Re}\left[\alpha_i + \min_{1\le j\le d} u_i\left(\frac{b_j}{B_j}\right)\right] > 0$$

$$|\arg(z)| < \frac{1}{2}\phi\pi$$

For j = 1,…,l, i=1,…,n and $n \in N$.

(2) 2 ND Inequality:-

$$I_{p+1,q+n}^{d+n,u}\left[z\left|\begin{matrix}\{a_p,A_p\},(1+a_i,u_i)_{i=i,n}\\\left(\beta+\sum_{i=1}^{n}a_i;u+\sum_{i=1}^{n}(u_i)\right),\{b_q,B_q\}\end{matrix}\right.\right]$$

$$\le I_{p+1,q+n}^{d+n,u}\left[z\left|\begin{matrix}\{a_p,A_p\},(1+a_ix,u_ix)_{i=i,n}\\\left(\beta+\sum_{i=1}^{n}a_ix;u+\sum_{i=1}^{n}(u_i)x\right),\{b_q,B_q\}\end{matrix}\right.\right]$$

$$\le I_{p+1,q}^{d,u}\left[z\left|\begin{matrix}\{a_p,A_p\}\\(\beta,u),\{b_q,B_q\}\end{matrix}\right.\right] \tag{2.2}$$

Provided that;

$x \in [0,1),$

$$\operatorname{Re}\left[a_i - \min_{1\le j\le d} u_i\left(\frac{b_j}{B_j}\right)\right] > 0$$

$$\operatorname{Re}\left[\beta - \min_{1\le j\le d} u\left(\frac{b_j}{B_j}\right)\right] > 0$$

$$|\arg(z)| < \frac{1}{2}\varphi\pi$$

For j = 1,…,l, i=1,…,n and $n \in N$.

(3) 3rd Inequality:-

$$I_{p+n,q+2}^{d,u+n}\left[z\left|\begin{matrix}(-a_i,u_i)_{i=i,n},\\\left(\beta+\sum_{i=1}^{n}a'_i;u+\sum_{i=1}^{n}(u_i)\right),\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}\{a_p,A_p\}\\\{b_q,B_q\},\left(-\beta-\sum_{i=1}^{n}a'_i;u+\sum_{i=1}^{n}(u_i)\right)\end{matrix}\right|\right]$$

$$\le I^{d,u+n}_{p+n,q+2}\left[z\left|\begin{matrix}\left(-a_i x, xu_i\right)_{i=i,n},\\ \left(\beta + x\sum_{i=1}^{n} a_i' ; x\left(u+\sum_{i=1}^{n}\left(u_i\right)\right)\right), \\ \{a_p, A_p\} \\ \{b_q, B_q\}, \left(-\beta + x\sum_{i=1}^{n}\left(u_i\right)\right)\end{matrix}\right.\right]$$

$$\le I^{d,u+n}_{p+n,q+2}\left[z\left|\begin{matrix}\{a_p, A_p\}\\ (\beta, u), \{b_q, B_q\}\end{matrix}\right.\right] \tag{2.3}$$

Provided that; $x \in [0,1)$,

$$\operatorname{Re}\left[\beta - \min_{1\le j\le d} u\left(\frac{b_j}{B_j}\right)\right] > 1$$

$$\operatorname{Re}\left[a_i - \min_{1\le j\le d} u_i\left(\frac{b_j}{B_j}\right)\right] > 0$$

$$\left|\arg(z)\right| < \frac{1}{2}\varphi\pi$$

$\beta \ge 1$, $a_i > 0$, For j = 1,…,l, i=1,…,n and $n \in N$.

(4) 4th Inequality:-

$$I^{d+1,u}_{p+1,q+n}\left[z\left|\begin{matrix}\{a_p, A_p\}, \left(1+a_i, u_i\right)_{i=i,n}\\ \{b_q, B_q\}, \left(-\beta - \sum_{i=1}^{n} a_i ; u + \sum_{i=1}^{n}\left(u_i\right)\right)\end{matrix}\right.\right]$$

$$\le I^{d+1,u+n}_{p+n,q+1}\left[z\left|\begin{matrix}\{a_p, A_p\}, \left(1+a_i x, u_i\right)_{i=i,n}\\ \{b_q, B_q\}, \left(-\beta - x\sum_{i=1}^{n} a_i ; x\left(u + \sum_{i=1}^{n}\left(u_i\right)\right)\right)\end{matrix}\right.\right]$$

$$\le I^{d,u}_{p,q+1}\left[z\left|\begin{matrix}\{a_p, A_p\}\\ \{b_q, B_q\}, (1-\beta, u)\end{matrix}\right.\right] \tag{2.4}$$

Provided that; $x \in [0,1)$,

$$\operatorname{Re}\left[a_i - \min_{1\le j\le d} u_i\left(\frac{b_j}{B_j}\right)\right] > 0$$

$$\operatorname{Re}\left[\beta + \min_{1\le j\le d} u\left(\frac{b_j}{B_j}\right)\right] > 1$$

$$\left|\arg(z)\right| < \frac{1}{2}\varphi\pi$$

For j = 1,…,l, i=1,…,n and $n \in N$.

Proof:-

To establish (2.1) replacing the I-function in the inequality as a Mellin-Barnes type of counter integral which is permissible due to absolute convergence of involved in the process. We have,

$$I^{M,N}_{P,Q}[z] = \int \frac{\prod_{j=1}^{M}\Gamma^{\beta_j}\left(b_j - B_j s\right)\prod_{j=1}^{N}\Gamma^{\alpha_j}\left(1-a_j + A_j s\right)}{\prod_{j=M+1}^{Q}\Gamma^{\beta_j}\left(1-b_j + B_j s\right)\prod_{j=N+1}^{P}\Gamma^{\alpha_j}\left(a_j - A_j s\right)} \times$$

$$\left\{\begin{matrix}\dfrac{\Gamma\left(1-(-\alpha_i)+u_i s\right) i=1,n}{\Gamma\left(1-\left(-\beta-\sum_{i=1}^{n} a_i + (u+u_i)s\right)\right)}\\ \le \dfrac{\Gamma\left(1-(-\alpha_i x)+u_i x s\right) i=1,n}{\Gamma\left(\beta + us + x\sum_{i=1}^{n}\left(\alpha_i + u_i s\right) i=1,n\right)}\\ \le \dfrac{1}{\Gamma\left(1-(1-\beta)+us\right)}\end{matrix}\right\} z^s ds$$

Now evaluating the inner inequality with the help of result Nguyen Van Vins and Ngo Phuoc Nguyen Ngoc [3], and interpreting the result. Thus obtained with the help of (1.1), we obtain the RHS of (2.1).

3. CONCLUSION

The result involved in the research paper are quite general nature. So many known and unknown results can be obtained from it.

REFERENCES

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