Large Deflection of A Circular Plate and $\widetilde{H}$-Function

Vijay Kumar Singhal
Department of Mathematics, Swami Keshvanand Institute of Technology, Management & Gramothan, Jaipur, 302017 (INDIA)
Email-
Received 18.02.2019 received in revised form 25.04.2019, accepted 30.04.2019

Abstract: The aim of the present paper is to find the bending stresses and deflections for a clamped circular plate under non-uniform load. The load shape is assumed as a function involving Jacobi polynomials, H̃-function and Srivastava polynomials. The deflection is obtained as a convergent infinite series. The small deflection is obtained as a special case.

Keywords: **H̃**-function, A general class of polynomials, Jacobi polynomials, Large and small deflection, Bending stresses.

1. INTRODUCTION

In 1972, Srivastava polynomials [1] defined as:

$$S_n^m[x] = \sum_{k=0}^{[\frac{n}{m}]} (-n)_{mk} A_{n,k} \frac{x^k}{k!},\ n = 0,1,2, \quad .. (1)$$

where m is an arbitrary positive integer and the coefficients $A_{N,k}\ (N,k \geq 0)$ are arbitrary constants, real or complex.

In 1987, Hussain[2,3] defined and studied the $\widetilde{H}$ -function as:

$$\widetilde{H}_{P,Q}^{M,N}[z] = \widetilde{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j,\alpha_j;\tau_j)_{1,N}(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}(b_j,\beta_j;\zeta_j)_{M+1,Q} \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Psi(\xi) z^{\xi} d\xi \quad \ldots(2)$$

where

$$\Psi(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{\tau_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{\zeta_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \quad (3)$$

for the convergence and existence condition, basic properties of $\widetilde{H}$- function, one may refer to the work by by Buschman and Srivastava [4].

In the present paper the large deflection of a clamped circular plate under non-uniform load following Berger's approximation [5] the plane displacement and the bending stresses for the circular plate are obtained. Applied external pressure p is assumed to be axis symmetric. The pressure $p(r)$ is taken as a function involving Jacobi polynomials, Srivastava polynomials, $\widetilde{H}$-function as

$$p(r) = C_0\left(1-\frac{r^2}{\rho^2}\right)^{\alpha} P_\beta^{a,b}\left(1-\frac{2r^2}{\rho^2}\right) S_n^m\left(1-\frac{r^2}{\rho^2}\right)\widetilde{H}_{P,Q}^{M,N}\left(1-\frac{r^2}{\rho^2}\right) \quad \ldots(4)$$

where $P_\beta^{a,b}(y)$ is the well known Jacobi polynomials [6] and C_0 is the arbitrary constant.

2. STATEMENT OF PROBLEM

The approximate equations for a clamped circular plate of flexural rigidity D, Thickness t and radius ρ, undergoing large deflection due to an externally applied non uniform load $p(r)$ following Berger's approximation method [5], may be written as:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-V^2w\right) = \frac{p}{D} = \psi(r) \ldots(5)$$

where V is a normalized constant of integration given by

$$\frac{dx}{dr}+\frac{x}{r}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2 = \frac{V^2t^2}{12}, \quad \ldots(6)$$

where x is the radial displacement and w is the plate deflection normal to the middle plane of the clamped circular plate.

The problem possesses following boundary conditions

$$w = \frac{dw}{dr} = 0 \quad at\ r = \rho$$
$$x = 0 \quad at\ r = \rho \quad \ldots(7)$$

3. SOLUTION OF THE PROBLEM

Let us take

$$w = \sum_i A_j[J_0(r\lambda_i) - J_0(\rho\lambda_i)] \quad \ldots(8)$$

where λ_i being the i^{th} root of $J_1(\rho\lambda_i) = 0$.

The boundary conditions (7) are satisfied by above equation.

For the value of w given by equation (8), the equation (5) becomes

$$\sum_i A_i\lambda_i^2(V^2+\lambda_i^2)J_0(r\lambda_i) = \Phi(r) \quad \ldots (9)$$

by expanding $\Phi(r)$ in terms of Bessel function and then integrating we get

$$\int_0^\rho \sum_i A_i\lambda_i^2(V^2+\lambda_i^2)J_0^2(r\lambda_i)\,rdr = \int_0^\rho \Phi(r)J_0(r\lambda_i)rdr$$

or

$$\frac{1}{2}A_i\rho^2\lambda_i^2(V^2+\lambda_i^2)J_0^2(\rho\lambda_i) = \int_0^\rho r\Phi(r)J_0(r\lambda_i)dr$$

Hence $A_i = \frac{2\int_0^\rho r\Phi(r)J_0(r\lambda_i)dr}{\rho^2\lambda_i^2(V^2+\lambda_i^2)J_0^2(\rho\lambda_i)}$ …(10)

Using Erdélyi[7], Equations (1), (2),(3) and the definition of Bessel function and then interchanging the order of summation and integration, we get an interesting integral

$$\int_0^1 u^{2\tau+1}(1-u^2)^\alpha P_\beta^{a,b}(1-2u^2)S_n^m(1-u^2)\tilde{H}_{P,Q}^{M,N}(1-u^2)J_\mu(uv)du =$$

$$\sum_{l=0}^{[\frac{n}{m}]}\frac{(-n)_{ml}}{l!}A_{n,l}\sum_{l'=0}^{\infty}\sum_{l''=0}^{\infty}\frac{(-1)^{l'}(-\beta)_{l''}}{\beta!\,l'!\,l''!}$$

$$.\frac{(1+a+b+\beta)_{l''}(1+\tau+\frac{\lambda}{2}+s)_{l''}}{\Gamma(1+a+l'')}\frac{\Gamma(1+a+\beta)}{\Gamma(\lambda+l'+1)}\left(\frac{v}{2}\right)^{\lambda+2l'}$$

$$.\tilde{H}_{P+1,Q+1}^{M,N+1}\left[1\left|\begin{matrix}(-\alpha-l,1;1)\\(b_j,\beta_j)_{1,M}(b_j,\beta_j:\zeta_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j;\tau_j)_{1,N}(a_j,\alpha_j)_{N+1,P}\\(-1-\alpha-l-l'-l''-\tau-\frac{\lambda}{2},1:1)\end{matrix}\right] \quad \ldots(11)$$

Where

$Re(a) > -1,\quad Re(b) > -1,\quad Re(\tau) > -1,\quad Re(\alpha) > -1,$

$Re(\lambda) > -\frac{1}{2},\ Re\left(\alpha+\frac{b_j}{\beta_j}\right) > 0, j = 1,2,\ldots,m$,

By equations (10) and (11) we have

$$A_i = \frac{C_0}{D}\frac{\Gamma(1+a+\beta)}{\beta!\,\lambda_i^2(V^2+\lambda_i^2)J_0^2(\rho\lambda_i)}$$

$$.\sum_{l=0}^{[\frac{n}{m}]}\frac{(-n)_{ml}}{l!}A_{n,l}\sum_{l'=0}^{\infty}\sum_{l''=0}^{\infty}\frac{(-1)^{l'}(-\beta)_{l''}}{l'!\,l''!}$$

$$.\frac{(1+a+b+\beta)_{l''}(1+l')_{l''}}{\Gamma(1+a+l'')}\left(\frac{\rho\lambda_i}{2}\right)^{2l'}$$

$$.\tilde{H}_{P+1,Q+1}^{M,N+1}\left[1\left|\begin{matrix}(-\alpha-l,1;1)\\(b_j,\beta_j)_{1,M}(b_j,\beta_j:\zeta_j)_{M+1,Q}\\(a_j,\alpha_j;\tau_j)_{1,N}(a_j,\alpha_j)_{N+1,P}\\(-1-\alpha-l-l'-l'',1:1)\end{matrix}\right.\right] \quad \ldots(12)$$

By equations (12) and (8), we obtain

$w = R_1\sum_i\frac{R_2}{(V^2+\lambda_i^2)}[J_0(r\lambda_i)-J_0(\rho\lambda_i)]$ …(13)

where $R_1 = \frac{C_0}{D}\frac{\Gamma(1+a+\beta)}{\beta!}$

and

$$R_2 = \frac{1}{\lambda_i^2J_0^2(\rho\lambda_i)}.\sum_{l=0}^{[\frac{n}{m}]}\frac{(-n)_{ml}}{l!}A_{n,l}\sum_{l'=0}^{\infty}\sum_{l''=0}^{\infty}\frac{(-1)^{l'}(-\beta)_{l''}}{l'!l''!}$$

$$.\frac{(1+a+b+\beta)_{l''}(1+l')_{l''}}{\Gamma(1+a+l'')}\left(\frac{\rho\lambda_i}{2}\right)^{2l'}$$

$$.\tilde{H}_{P+1,Q+1}^{M,N+1}\left[1\left|\begin{matrix}(-\alpha-l,1;1)\\(b_j,\beta_j)_{1,M}(b_j,\beta_j:\zeta_j)_{M+1,Q}\\(a_j,\alpha_j;\tau_j)_{1,N}(a_j,\alpha_j)_{N+1,P}\\(-1-\alpha-l-l'-l'',1:1)\end{matrix}\right.\right]$$

The radial displacement is

$$rx = \frac{V^2t^2r^2}{24}-\frac{1}{2}\sum_{l'=1}^{\infty}A_i^2\lambda_i^2\left[\frac{r^2}{2}\left\{J_1'^2(r\lambda_i)+\left(1-\frac{1}{r^2\lambda_i^2}\right)J_1^2(r\lambda_i)\right\}\right]-\frac{1}{2}\sum_{l'=1}^{\infty}\sum_{\substack{j=1\\l'\neq j}}^{\infty}A_iA_j\lambda_i\lambda_j.\left[\frac{\lambda_iJ_2(r\lambda_i)J_1(r\lambda_j)-\lambda_jJ_2(r\lambda_j)J_1(r\lambda_i)}{\lambda_i^2-\lambda_j^2}\right]+C' \quad \ldots(14)$$

where C' is the constant of integration.

Equation (14) can be obtained by using equations (6) and (8).

On applying the conditions $u = 0$ at $r = \rho$ and using $J_1(\rho\lambda_i) = 0$, we obtain

$$C' = -\frac{V^2t^2\rho^2}{24}+\frac{1}{4}\sum_{l'=1}^{\infty}A_i^2\lambda_i^2\rho^2J_0^2(\rho\lambda_i)$$

When $V = 0$, the differential equation (5) corresponds to the small deflection equation. By virtue of the equation (13), we get

$w = R_1\sum_i\frac{R_2}{\lambda_i^2}[J_0(r\lambda_i)-J_0(\rho\lambda_i)$ …(15)

By taking $r = 0$ in the equation (13), the deflection at the centre of plate is obtained as

$w_0 = R_1\sum_i\frac{R_2}{(V^2+\lambda_i^2)}[1-J_0(\rho\lambda_i)]$ …(16)

For the small deflection at centre of the plate, we obtain

$w_0 = R_1\sum_i\frac{R_2}{\lambda_i^2}[1-J_0(\rho\lambda_i)]$ …(17)

Berger, H. M. [5] introduced the bending stresses at the surface of the circular plate as

$\sigma_r = -\frac{6D}{t^2}\left(\frac{d^2w}{dr^2}+\frac{v}{r}\frac{dw}{dr}\right)$ …(18)

$\sigma_\theta = -\frac{6D}{t^2}\left(v\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)$ …(19)

here υ is the Poisson's ratio.

The deflection obtained in equation (13) can be used to determine these stresses. Hence the bending stresses at the centre are

$$(\sigma_r)_{r=0} = (\sigma_\theta)_{r=0} = \frac{3D}{t^2}R_1\sum_i\frac{R_2}{(V^2+\lambda_i^2)}[\upsilon+1] \quad \ldots(20)$$

Also the bending stresses at the edge are

$(\sigma_r)_{r=\rho} = \frac{6D}{t^2} R_1 \sum_i \frac{R_2}{(V^2+\lambda_i^2)} \lambda_i^2 J_0(\rho\lambda_i)$ …(21)

$(\sigma_\theta)_{r=\rho} = \frac{6D}{t^2} R_1 \sum_i \frac{R_2}{(V^2+\lambda_i^2)} \upsilon\lambda_i^2 J_0(\rho\lambda_i)$ …(22)

4. BEHAVIOUR OF THE FAMILY OF LOAD SHAPE *p*(*r*)

By (1.4), we have

$p(r) = C_0\left(1-\frac{r^2}{\rho^2}\right)^{\alpha} P_\beta^{a,b}\left(1-\frac{2r^2}{\rho^2}\right) S_n^m\left(1-\right.$ *r2ρ2HP,QM,N1−r2ρ2*

$$= C_0 \sum_{l=0}^{[\frac{n}{m}]} \frac{(-n)_{ml}}{l!} A_{n,l} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Psi(s)\left(1-\frac{r^2}{\rho^2}\right)^{\alpha+l+s} \cdot {}_2F_1\left[{-\beta, a+b+\beta+1 \atop a+1}; \frac{r^2}{\rho^2}\right] ds \quad \ldots(23)$$

Following two cases have been considered here

(i) On taking $\beta = 1$ and $a, b, \alpha > 0$ then

$$p(r) = C_0 \sum_{l=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{ml}}{l!} A_{n,l} \frac{1}{2\pi i}$$

$$\cdot\int_{-i\infty}^{i\infty} \Psi(s)\left(1-\frac{r^2}{\rho^2}\right)^{\alpha+l+s}\left(1-\frac{(a+b+2)}{1+a}\frac{r^2}{\rho^2}\right) ds \quad \ldots(24)$$

At $r = \rho\sqrt{\left(\frac{1+a}{a+b+2}\right)}$ and $r = \rho$, we have $p(r) = 0$

It shows the dependency of *r* on a and b and indicates that load shape $p(r)$ changes sign at

$r = \rho\sqrt{\left(\frac{1+a}{a+b+2}\right)}$ as is clear from (25).

(ii) On taking $\beta = a = b = 0, n = 0, \alpha > 0,$

$\tau_j = \zeta_j = 1, P = N = 0, M = Q = 1; b_j = a_j = \alpha_j = 0;\ \beta_j = 1$

we have

$p(r) = C_0 \sum_n \frac{(-1)^n}{n!}\left(1-\frac{r^2}{\rho^2}\right)^{n+\alpha}$ …(26)

For suitable values of *n* it is an axially symmetric distribution of the force over the plate acting in the positive direction. We also have a number of force distributions of different intensities for various values of α.

5. CONCLUSION

As the single function *p*(*r*) represents several types of loading and therefore the result obtained in the present study is capable of unifying scores of hitherto scattered results in the concerned literature.

REFERENCES

[1] Srivastava H.M., A contour integral involving Fox's H-function, Indian J. Math.,Vol-14, pp.1-6 (1972).

[2] Hussian A. A. I., New properties of hypergeometric series derivable from Feynman integrals: I, J. Phys. A: Math. Gen., Vol-20, pp.4109-4117 (1987).

[3] Hussian A. A. I., New properties of hypergeometric series derivable from Feynman integrals

[4] Berger H.M., A new approach to the analysis of Large deflection of Plates, Jour. Appl. Mech. Trans. ASME, Vol.-22, pp.465-472 (1955)

[5] Szego G., Orthogonal polynomials, American mathematical society, Colloquium Publications, Vol.-23, revised edition (1959)

[6] Erdélyi A., Table of Integral Transforms, Vol.-2, Mcgraw-Hill, New York, (1954).