# Application of Laplace Decomposition Method to Fractional Riccati Equations

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**Abstract- In this manuscript, we apply a new technique named the Laplace Decomposition Method (LDM) to the fractional differential equation called the Riccati equation. Laplace Decomposition Method (LDM) is based on the Laplace Transform Method (LTM) and Adomain Decomposition Method (ADM). We attempt to give an estimated solution to the fractional Riccati differential equation using Laplace decomposition method and we also observe the behavior of the solution obtained. LDM makes it very easy to solve linear and non-linear fractional differential equations and gives exact solutions in the form of convergence series. The graphical interpretation of the behavior of the result is also given at the end of this manuscript, which is comparable with the results obtained by other methods.**

**Keywords–** Fractional Riccati Equations, Laplace Decomposition Method, Adomian Decomposition Method.

## 1. INTRODUCTION

Analytical and numerical answers to fractional-order differential equations (FDE) have always drawn the interest of researchers due to having applications in several areas of pure and applied sciences, engineering, and biological sciences also.

The Italian nobleman Count Jacopo Francesco Riccati after whose name the Riccati differential equation is named and gained popularity. The book of Reid [9] describes the main theories of Riccati equation, with implementations to random processes, optimal control, and diffusion problems [1,3]. Fractional Riccati differential equations is of use in various fields, although discussions on the numerical methods for these equations are not very common. Odibat and Momani [7] explored a modified homotopy perturbation method for fractional Riccati differential equations. Khader [3] researched the fractional Chebyshev finite difference method for fractional Riccati differential equations. Li [5] have resolved this problem by applying quasi-linearization technique.

In particular, the Fractional Riccati equation [1], being discussed here is represented as

$$D^{\gamma} y(\varepsilon) = P(\varepsilon) + Q(\varepsilon)y(\varepsilon) + R(\varepsilon)y^2(\varepsilon), \epsilon \in R, 0 < \gamma \le 1, \varepsilon > 0 \quad \text{(i)}$$

where $y^{(\propto)}(0) = g_{\propto}$, $\propto = 0,1,2.\dots.n-1$; $P(\varepsilon)$, $Q(\varepsilon)$and $R(\varepsilon)$ are given functions, $g_{\propto}(\propto = 0,1,2.\dots.n-1)$, are random constants and $\gamma$ is the order of fractional derivative.

The main of this paper is to save the fractional Riccati equation by using the Laplace decomposition method. Through this work, we can enhance the applications of the Laplace decomposition method and fractional Riccati equation.

## 2. PRELIMINARIES AND NOTATIONS

***Definition 1*** The fractional-order Riemann-Liouville derivative of the function $y(\varepsilon)$ for $\varepsilon$ and order $\gamma$> 0 is defined as [6]

$$D_{\varepsilon}^{\gamma} y(\varepsilon) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{d\varepsilon^n} \int_0^t (\varepsilon)^{n-\gamma-1} y(u)du, \quad n-1 < \gamma \le n. \quad \text{(ii)}$$

***Definition 2*** The fractional-order Caputo derivative of the function $y(\varepsilon)$ for $\varepsilon$ and order $\gamma$> 0 is defined as [6]

$$D_{\varepsilon}^{\gamma} y(\varepsilon) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (\varepsilon)^{n-\gamma-1} y^n(u)du, \quad n-1 < \gamma \le n. \quad \text{(iii)}$$

***Definition 3*** The Laplace transform of $D_{\varepsilon}^{\gamma} y(\varepsilon)$ is defined as [5,6]

$$L\left[D_{\varepsilon}^{\gamma} y(\varepsilon)\right] = s^{\gamma} L[y(\varepsilon)] - \sum_{\theta=0}^{\propto-1} s^{\gamma-\theta-1}\, y^{(\theta)}(0), \quad \propto -1 < \gamma \le \propto \quad \text{(iv)}$$

## 3. LAPLACE DECOMPOSITION METHOD (LDM)

The Laplace Transform and Adomian decomposition method together form the Laplace Decomposition Method (LDM) [10]. It is a mathematical tool that can be applied to solve linear, nonlinear, ordinary, and partial differential equations and fractional-order nonlinear differential equations also [2,4,11].

To illustrate LDM, consider the equation

$$D_\varepsilon^\gamma y(\varepsilon) + Py(\varepsilon) + Qy(\varepsilon) = 0 \qquad \text{(v)}$$

where $D_\varepsilon^\gamma y(\varepsilon)$ is the fractional-order Caputo derivative of order $\gamma$, $\propto -1 < \gamma \leq \propto, \propto \in N$ , P is the linear operator and Q is the non-linear operator and

$$y(\varepsilon) = y(0) \text{ at } \varepsilon = 0 \qquad \text{(vi)}$$

Taking Laplace transform of (v), we have

$$L\left(D_\varepsilon^\gamma y(\varepsilon) + Py(\varepsilon) + Qy(\varepsilon)\right) = 0$$

which because of (vi) gives

$$L[y(\varepsilon)] = \frac{1}{s^\gamma}\sum_{n=0}^{\propto-1} s^{\gamma-\theta-1}y^\theta(0) - \frac{1}{s^\gamma}L[Py(\varepsilon) + Qy(\varepsilon)], \qquad \text{(vii)}$$

Now taking inverse Laplace transform of (vii) we get

$$y(\varepsilon) = L^{-1}\left\{\frac{1}{s^\gamma}\sum_{n=0}^{\propto-1} s^{\gamma-\theta-1}y^\theta(0) - \frac{1}{s^\gamma}L[Py(\varepsilon) + Qy(\varepsilon)]\right\}, \qquad \text{(viii)}$$

By using the initial condition, we arrive at

$$y(\varepsilon) = y(0) - L^{-1}\left\{\frac{1}{s^\gamma}L[Py(\varepsilon) + Qy(\varepsilon)]\right\} \qquad \text{(ix)}$$

Now we represent the solution of (ix) in terms of an infinite series as

$$y(\varepsilon) = \sum_{i=0}^{\infty} y_i(t) = y_0(t) + y_1(t) + y_2(t) + \cdots + y_i(t) + \cdots \qquad \text{(x)}$$

here $Qy(\varepsilon) is\ a$ nonlinear operator represented as $Qy(\varepsilon) = \sum_{i=0}^{\infty} A_i$ be (xi)

and Adomian polynomial $A_i$ is defined as

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\delta^i}\left\{Q\sum_{i=0}^{\infty}(\delta^i y_i)\right\}\right]_{\varepsilon=0}, \quad i = 0,1,2,\ldots, \qquad \text{(xii)}$$

where

$$y_0(t) = y(0), \qquad \text{(xiii)}$$

$$y_1(t) = -L^{-1}\left\{\frac{1}{s^\gamma}L[Py_0(t) + A_0]\right\}, \qquad \text{(xiv)}$$

and $y_{i+1}(t) = -L^{-1}\left\{\frac{1}{s^\gamma}L[Py_i(t) + A_i]\right\}$, $i \geq 1$. (xv)

## 4. MAIN RESULTS

In this section we consider the following two problems to illustrate the application of LDM to obtain the solution to Fractional Riccati differential equations:

***Problem 1:*** Consider the Fractional Riccati differential equation

$$D_\varepsilon^\gamma y(\varepsilon) = -y^2(\varepsilon) + 1, \quad 0 < \varepsilon \leq 1 \qquad \text{(xvi)}$$

with primary condition

$$y(0) = y_0 = 0 \qquad \text{(xvii)}$$

Taking Laplace transform of (xvi) and then using (xvii), we get

$$L[y(\varepsilon)] = \frac{1}{s^{\gamma+1}} - \frac{1}{s^\gamma}L[y^2(\varepsilon)] \qquad \text{(xviii)}$$

which on taking inverse Laplace transform gives

$$y(\varepsilon) = L^{-1}\left\{\frac{1}{s^{\gamma+1}} - \frac{1}{s^\gamma}L[y^2(\varepsilon)]\right\} \qquad \text{(xix)}$$

Now, on applying Laplace Decomposition Method, we arrive at

$$y_0(\varepsilon) = y(0) = \frac{\varepsilon^\gamma}{\Gamma[\gamma+1]} \qquad \text{(xx)}$$

$$y_1(\varepsilon) = -L^{-1}\frac{1}{s^\gamma}\left(L[{y_0}^2(\varepsilon)]\right)$$

$$y_1(\varepsilon) = -\frac{\Gamma[2\gamma+1]\varepsilon^{3\gamma}}{\left[\Gamma[\gamma+1]\right]^2\Gamma[3\gamma+1]} \qquad \text{(xxi)}$$

$$y_2(\varepsilon) = -L^{-1}\left(\frac{1}{s^\gamma}L[2y_0y_1]\right)$$

$$y_2(\varepsilon) = \frac{2\Gamma[2\gamma+1]\Gamma[4\gamma+1]\varepsilon^{5\gamma}}{\left[\Gamma[\gamma+1]\right]^3\Gamma[3\gamma+1]\Gamma[5\gamma+1]} \qquad \text{(xxii)}$$

Thus, the solution can be expressed in the series form as

$$y(\varepsilon) = y_0(\varepsilon) + y_1(\varepsilon) + y_2(\varepsilon) + \cdots + y_i(\varepsilon) + \cdots = \frac{\varepsilon^\gamma}{\Gamma[\gamma+1]} - \frac{\Gamma[2\gamma+1]\varepsilon^{3\gamma}}{\left[\Gamma[\gamma+1]\right]^2\Gamma[3\gamma+1]} + \frac{2\Gamma[2\gamma+1]\Gamma[4\gamma+1]\varepsilon^{5\gamma}}{\left[\Gamma[\gamma+1]\right]^3\Gamma[3\gamma+1]\Gamma[5\gamma+1]} + \ldots \qquad \text{(xxiii)}$$

which on taking $\gamma = 1$, gives

$$y(\varepsilon) = \lim_{n\to\infty} y_n(\varepsilon) = \varepsilon - \frac{\varepsilon^3}{3} + \frac{2}{15}\varepsilon^5 + \ldots \qquad \text{(xxiv)}$$

***Problem 2:*** Consider the Fractional Riccati differential equation

$$D_\varepsilon^\gamma y(\varepsilon) = 2y(\varepsilon) - y^2(\varepsilon) + 1, \quad 0 < \varepsilon \leq 1 \qquad \text{(xxv)}$$

with primary condition

$$y(0) = y_0 = 0 \qquad \text{(xxvi)}$$

Taking Laplace transform of (xxv) and then using (xxvi), we get

$$L[y(\varepsilon)] = \frac{1}{s^{\gamma+1}} + \frac{1}{s^\gamma}L[2y(\varepsilon) - y^2(\varepsilon)] \qquad \text{(xxvii)}$$

which on taking inverse Laplace transform gives

$$y(\varepsilon) = L^{-1}\left\{\frac{1}{s^{\gamma+1}} + \frac{1}{s^\gamma}L[2y(\varepsilon) - y^2(\varepsilon)]\right\} \qquad \text{(xxviii)}$$

Now, on applying Laplace Decomposition Method, we arrive at

$$y_0(\varepsilon) = y(0) = \frac{\varepsilon^{\gamma}}{\Gamma[\gamma+1]} \tag{xxix}$$

$$y_1(\varepsilon) = L^{-1}\left(\frac{1}{s^{\gamma}} L[2y_0(\varepsilon) - y_0{}^2(\varepsilon)]\right)$$

$$y_1(\varepsilon) = 2\frac{\varepsilon^{2\gamma}}{\Gamma[2\gamma+1]} - \frac{\Gamma[2\gamma+1]\varepsilon^{3\gamma}}{\left[\Gamma[\gamma+1]\right]^2 \Gamma[3\gamma+1]} \tag{xxx}$$

$$y_2(\varepsilon) = -L^{-1}\left(\frac{1}{s^{\gamma}} L[2y_1 - 2y_0 y_1]\right)$$

$$y_2(\varepsilon) = 4\frac{\varepsilon^{3\gamma}}{\Gamma[3\gamma+1]} - \frac{2\Gamma[2\gamma+1]\varepsilon^{4\gamma}}{\left[\Gamma[\gamma+1]\right]^2\Gamma[4\gamma+1]} - \frac{4\Gamma[3\gamma+1]\varepsilon^{4\gamma}}{\Gamma[\gamma+1]\Gamma[2\gamma+1]\Gamma[4\gamma+1]} + \frac{2\Gamma[2\gamma+1]\Gamma[4\gamma+1]\varepsilon^{5\gamma}}{\left[\Gamma[\gamma+1]\right]^3\Gamma[3\gamma+1]\Gamma[5\gamma+1]} \tag{xxxi}$$

Thus, the solution can be expressed in the series form as

$$\begin{aligned} y(\varepsilon) &= y_0(\varepsilon) + y_1(\varepsilon) + y_2(\varepsilon) + \cdots + y_i(\varepsilon) + \cdots \\ &= \frac{\varepsilon^{\gamma}}{\Gamma[\gamma+1]} + 2\frac{\varepsilon^{2\gamma}}{\Gamma[2\gamma+1]} - \frac{\Gamma[2\gamma+1]\varepsilon^{3\gamma}}{\left[\Gamma[\gamma+1]\right]^2\Gamma[3\gamma+1]} + 4\frac{\varepsilon^{3\gamma}}{\Gamma[3\gamma+1]} \\ &- \frac{2\Gamma[2\gamma+1]\varepsilon^{4\gamma}}{\left[\Gamma[\gamma+1]\right]^2\Gamma[4\gamma+1]} - \frac{4\Gamma[3\gamma+1]\varepsilon^{4\gamma}}{\Gamma[\gamma+1]\Gamma[2\gamma+1]\Gamma[4\gamma+1]} \\ &+ \frac{2\Gamma[2\gamma+1]\Gamma[4\gamma+1]\varepsilon^{5\gamma}}{\left[\Gamma[\gamma+1]\right]^3\Gamma[3\gamma+1]\Gamma[5\gamma+1]} + \ldots \end{aligned} \tag{xxxii}$$

which on taking $\gamma = 1$, gives

$$y(\varepsilon) = \lim_{n\to\infty} y_n(\varepsilon) = \varepsilon + \varepsilon^2 + \frac{\varepsilon^3}{3} - \frac{2\varepsilon^4}{3} + \frac{2}{15}\varepsilon^5 + \ldots \tag{xxxiii}$$

## 5. GRAPHICAL REPRESENTATION

The graphical representation of the equations (xxiii) and (xxxii) are showing with figure 1 and figure 2 with different values of $\gamma$ and $\varepsilon$ are as under:

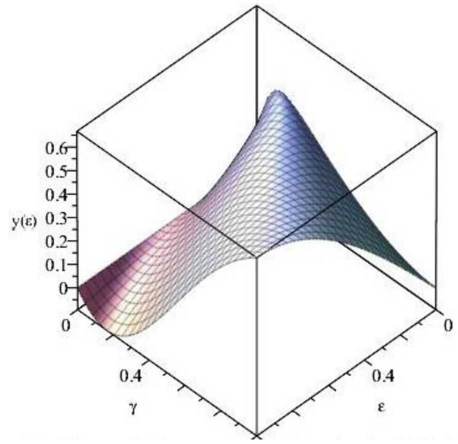


**Figure 1:** The Graphical representation of equation (xxiii) for different value of γ and ε

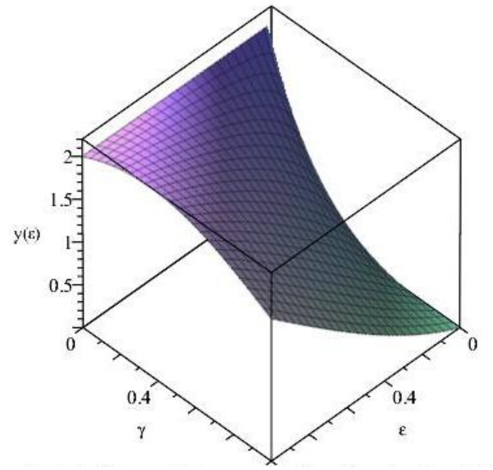


**Figure 2:** The Graphical representation of equation (xxxii) for different value of γ and ε

## 6. CONCLUSION

The present paper presents a novel approach to obtaining the solution of fractional order Riccati equation by applying the Laplace Decomposition Method (LDM). The graphical representations of the solutions obtained depict the behavior of the results.

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