A Novel Approach to Fractional Epidemic Model through Elzaki Decomposition Method

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Abstract- **This paper comprises the fractional order of epidemic model of a non-fatal syndrome from a population of constant size. The Elzaki transform is use to construct the analytical solution of the fractional order equation through maple software. An admirable comparative study has also been made with the previous published results through Graphs and Tables.**

Keywords– Epidemic model, Fractional differential equations, Elzaki Transform

1. INTRODUCTION

Epidemiology is the process to evaluate the origins of health-related consequences and illnesses in populations. There are three major epidemiology techniques as descriptive, analytic and experimental. McKendrik and Kermack [1] describe the well-known mathematical model to expecting the behavior of epidemic eruptions. This Model is known as SIR model of xyz model consists of three types at time τ.

$$\begin{cases} D^{\propto_1}x(\tau) = -\delta x(\tau)y(\tau) \\ D^{\propto_2}y(\tau) = \delta x(\tau)y(\tau) - \rho y(\tau) \\ D^{\propto_3}z(\tau) = \rho y(\tau) \end{cases}$$

s.t the initial condition

$$x(0) = n_1, y(0) = n_2, z(0) = n_3$$

Here $x(\tau)$ represent the number of susceptible entities, $y(\tau)$ represent the number of infected entities might be able to spread the diseases through contact with the susceptible, $z(\tau)$ represent the number of isolated entities who cannot come to be or communicate the disease for various reasons, δ represents the transition rate probability between the Susceptible and Infected, while ρ denotes the rate of transition between infected and recovered, also $0 \leq \delta, \rho \leq 1$.

Integer order epidemic model have been solved by many authors [2-5]. Fractional order epidemic model has been covenanted in [6-7].

The Adomian Decomposition method was first proposed by G. Adomian [8] in 1990. This method is being coupled with the Laplace Transform, natural decomposition method, Sumudu Transform and Elzaki transform techniques provides a remarkable concern in science and engineering [9-14].

In this paper we will discuss the Elzaki decomposition method for solving fractional order epidemic model. A comparative study has also been made with the previous published literature.

2. PRELIMINARIES AND NOTATIONS

2.1 Elzaki Decomposition Method (EDM):

Let us consider the following fractional non-linear differential equation to illustrate the EDM,

$$D_\tau^\propto X(\tau) + NX(\tau) + LX(\tau) = h(\tau), \quad m-1 < \propto \leq m, \quad m \in N, \tag{i}$$

where

$$X(\tau) = X(0) \text{ at } \tau = 0 \tag{ii}$$

where $D_\tau^\propto(.)$ is the fractional derivative of Caputo type, N is a linear operator and L shows a non-linear operator, with the derivatives of fractional order less than ∝.

Taking the Elzaki transform of (i)

$$E\left(D_\tau^\propto X(\tau) + NX(\tau) + LX(\tau)\right) = E(h(\tau))$$

by using (i)

$$E[X(\tau)] = \vartheta^\propto \sum_{k=0}^{m-1} \vartheta^{2-\propto-k} X^k(0) - \vartheta^\propto E[NX(\tau) + LX(\tau) - h(\tau)] \tag{iii}$$

taking inverse Elzaki transform

$$X(\tau) = H(\tau) - E^{-1}\{\vartheta^\propto E[NX(\tau) + LX(\tau)]\} \tag{iv}$$

where $H(\tau)$ is the term arising from the source term and the prescribed initial condition. The representation of the solution (iv) as an infinite series is given below

$$X(\tau) = \sum_{i=0}^{\infty} X_i(\tau) = X_0(\tau) + X_1(\tau) + X_2(\tau) + \cdots + X_i(\tau) + \cdots \tag{v}$$

let us decompose $LX(\tau)$ in Adomian polynomials as

$$LX(\tau) = \sum_{i=0}^{\infty} A_i \quad \text{(vi)}$$

where

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\varepsilon^i}\left\{L\sum_{i=0}^{\infty}(\varepsilon^i X_i)\right\}\right]_{\varepsilon=0}, \quad i = 0,1,2,..., \quad \text{(vii)}$$

$$\sum_{i=0}^{\infty} X_{i+1}(\tau) = H(\tau) - E^{-1}\{\vartheta^{\alpha} E[N\sum_{i=0}^{\infty} X_i(\tau) + \sum_{i=0}^{\infty} A_i]\}$$

where

$$X_0(\tau) = H(\tau) = E^{-1}\{\vartheta^{\alpha}\sum_{k=0}^{m-1}\vartheta^{2-\alpha-k}X^k(0) + \vartheta^{\alpha} E[h(\tau)]\} \quad \text{(viii)}$$

$$X_{i+1}(\tau) = -E^{-1}\{\vartheta^{\alpha} E[NX_i(\tau) + A_i]\}, \quad i \geq 1 \quad \text{(ix)}$$

3.MAIN RESULTS

Let us consider the fractional order Epidemic model with initial conditions

$$D^{\alpha_1}x(\tau) = -\delta x(\tau)y(\tau)$$
$$D^{\alpha_2}y(\tau) = \delta x(\tau)y(\tau) - \rho y(\tau), \text{ where } \alpha_1, \alpha_2, \alpha_3 > 0$$
$$D^{\alpha_3}z(\tau) = \rho y(\tau) \quad \text{(x)}$$

related to initial condition

$$x(0) = n_1, y(0) = n_2, z(0) = n_3$$

For this model the initial conditions are not independent, since they must satisfy the condition $n_1 + n_2 + n_3 = n$, where n is the total fixed number of the individuals in the given population. By applying Elzaki transform on equation (x)

$$E\{D^{\alpha_1}x(\tau)\} = E\{-\delta x(\tau)y(\tau)\}$$
$$E\{D^{\alpha_2}y(\tau)\} = E\{\delta x(\tau)y(\tau) - \rho y(\tau)\}$$
$$E\{D^{\alpha_3}z(\tau)\} = E\{\rho y(\tau)\} \quad \text{(xi)}$$

by using property of Elzaki transform

$$\frac{E(x)}{v^{\alpha_1}} - \frac{x(0)}{v^{\alpha_1-2}} = -\delta E\{x(\tau)y(\tau)\}$$
$$\frac{E(y)}{v^{\alpha_2}} - \frac{y(0)}{v^{\alpha_2-2}} = \delta E\{x(\tau)y(\tau)\} - \rho E\{\rho y(\tau)\}$$
$$\frac{E(z)}{v^{\alpha_3}} - \frac{z(0)}{v^{\alpha_3-2}} = \rho E\{y(\tau)\} \quad \text{(xii)}$$

by applying initial condition

$$E(x) = v^2 n_1 - \delta v^{\alpha_1} E\{x(\tau)y(\tau)\}$$
$$E(y) = v^2 n_2 + \delta v^{\alpha_2} E\{x(\tau)y(\tau)\} - \rho v^{\alpha_2} E\{y(\tau)\}$$
$$E(z) = v^2 n_3 + \rho v^{\alpha_3} E\{y(\tau)\} \quad \text{(xiii)}$$

the method assumes the solution as an infinite series

$$x = \sum_{k=0}^{\infty} x_k, \quad y = \sum_{k=0}^{\infty} y_k, \quad z = \sum_{k=0}^{\infty} z_k \quad \text{(xiv)}$$

$$xy = \sum_{k=0}^{\infty} A_k$$

where A_k (Adomian Polynomials) is given by

$$A_k = \frac{1}{k!}\left[\frac{d^k}{d\varepsilon^k}\left\{\sum_{i=0}^{k}(\varepsilon^i x_i)\sum_{i=0}^{k}(\varepsilon^i y_i)\right\}\right]_{\varepsilon=0} \quad \text{(xv)}$$

by putting equations (xiv) and (xv) in (xiii)

$$E(x_0) = v^2 n_1,$$
$$E(y_0) = v^2 n_2,$$
$$E(z_0) = v^2 n_3 \quad \text{(xvi)}$$

$$E(x_1) = -\delta v^{\delta n_1 n_2 1} E\{A_0\},$$
$$E(y_1) = \delta v^{\alpha_2} E\{A_0\} - \rho v^{\alpha_2} E\{y_0\},$$
$$E(z_1) = \rho v^{\alpha_3} E\{y_0\} \quad \text{(xvii)}$$

$$E(x_{k+1}) = -\delta v^{\alpha_1} E\{A_k\},$$
$$E(y_{k+1}) = \delta v^{\alpha_2} E\{A_k\} - \rho v^{\alpha_2} E\{y_k\},$$
$$E(z_{k+1}) = \rho v^{\alpha_3} E\{y_k\} \quad \text{(xviii)}$$

from equations (xvi), (xvii), and (xviii)

$$x_0 = n_1$$
$$y_0 = n_2$$
$$z_0 = n_3$$
$$x_1 = -\delta n_1 n_2 \frac{\tau^{\alpha_1}}{\Gamma(\alpha_1 + 1)}$$
$$y_1 = (\delta n_1 n_2 - \rho n_2)\frac{\tau^{\alpha_2}}{\Gamma(\alpha_2 + 1)}$$
$$z_1 = \rho n_2 \frac{\tau^{\alpha_3}}{\Gamma(\alpha_3 + 1)}$$
$$x_2 = (-\delta^2 n_1^2 n_2 + \delta\rho n_1 n_2)\frac{\tau^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2 + 1)} + (\delta^2 n_1 n_2^2)\frac{\tau^{2\alpha_1}}{\Gamma(2\alpha_1 + 1)}$$
$$y_2 = (\delta^2 n_1^2 n_2 - 2\delta\rho n_1 n_2 + \rho^2 n_2)\frac{\tau^{2\alpha_2}}{\Gamma(2\alpha_2+1)} - \delta^2 n_1 n_2^2 \frac{\tau^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$
$$z_2 = (\delta\rho n_1 n_2 - \rho^2 n_2)\frac{\tau^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3 + 1)}$$

we can write the solution

$$x(\tau) = x_0 + x_1 + x_2 + \ldots, y(\tau) = y_0 + y_1 + y_2 + \ldots, z(\tau) = z_0 + z_1 + z_2 \ldots \quad \text{(xix)}$$

4. NUMERICAL AND GRAPHICAL REPRESENTATION

Table-1 The number of susceptible entities $x(\tau)$ for various values of α, $n_1 = 20, n_2 = 15, n_3 = 10, \delta = 0.01, \rho = 0.02$

t	$\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$		$\alpha_1 = 1, \alpha_2 = 0.99, \alpha_3 = 0.99$		$\alpha_1 = 0.99, \alpha_2 = 0.99, \alpha_3 = 0.95$	
	Rida et al.[7]	Present study	Rida et al.[7]	Present study	Rida et al.[7]	Present study
0	15	15	15	15	15	15

1	17.6898	7.589114	17.7031	17.72134	17.7007	17.724681
2	20.2466	20.35791	20.2329	20.281496	20.2306	20.308423
3	22.5014	22.551978	22.4438	22.521090	22.4498	22.517960
4	24.2851	24.364791	24.1478	24.257131	24.2029	24.319465
5	25.4287	25.617895	25.2646	25.327954	25.3336	25.507314
6	25.7633	25.742180	25.5515	25.641831	25.6861	25.941431

Table-2 The number of infected entities $y(\tau)$ for various values of α, $n_1 = 20, n_2 = 15, n_3 = 10, \delta = 0.01, \rho = 0.02$

t	$\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$		$\alpha_1 = 1, \alpha_2 = 0.99, \alpha_3 = 0.99$		$\alpha_1 = 0.99, \alpha_2 = 0.99, \alpha_3 = 0.95$	
	Rida et al.[7]	Present study	Rida et al.[7]	Present study	Rida et al.[7]	Present study
0	20	20	20	20	20	20
1	16.9831	16.97364	16.9807	16.97245	16.9704	16.97236
2	14.0444	14.03231	14.0419	14.03156	14.0616	14.03161
3	11.3524	11.34128	11.3540	11.34014	11.4109	11.34102
4	9.07520	9.068974	9.07472	9.069012	9.17302	9.069123
5	7.38125	7.391561	7.40056	7.390436	7.50352	7.390462
6	6.43880	6.439127	6.46736	6.439264	6.55779	6.439291

Table-3 The number of isolated entities $z(\tau)$ for various values of α, $n_1 = 20, n_2 = 15, n_3 = 10, \delta = 0.01, \rho = 0.02$

t	$\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$		$\alpha_1 = 1, \alpha_2 = 0.99, \alpha_3 = 0.99$		$\alpha_1 = 0.99, \alpha_2 = 0.99, \alpha_3 = 0.95$	
	Rida et al.[7]	Present study	Rida et al.[7]	Present study	Rida et al.[7]	Present study
0	10	10	10	10	10	10
1	10.3271	10.519754	10.3289	10.519754	10.3348	10.437123
2	10.7090	10.961052	10.7079	10.961052	10.7019	10.984251
3	11.1462	11.087423	11.1112	11.087423	11.1129	11.412310
4	11.6397	11.681081	11.5610	11.681081	11.5701	11.951533
5	12.1900	12.340790	12.0521	12.340790	12.0744	12.363624
6	12.7979	12.920789	12.5841	12.920789	12.6265	12.901278

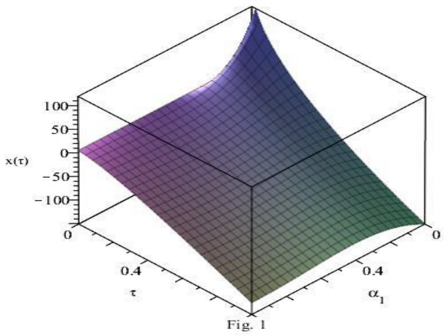

Figure-1: Variation of $x(\tau)$ for various values of α_1

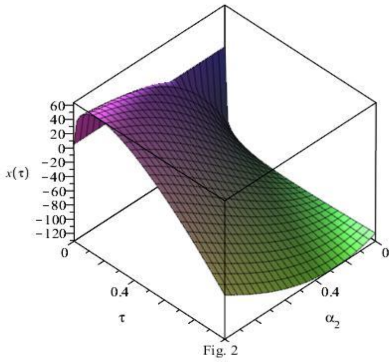

Figure-2: Variation of $x(\tau)$ for various values of α_2

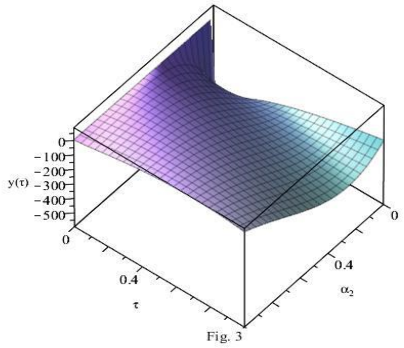

Figure-3: Variation of $y(\tau)$ for various values of α_2

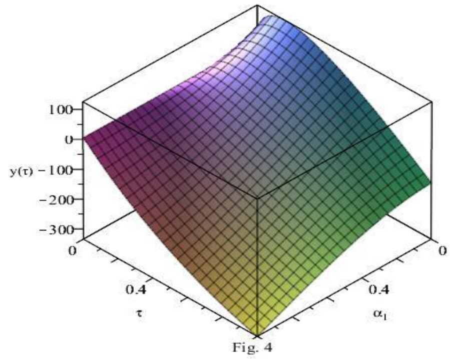

Figure-4: Variation of $y(\tau)$ for various values of α_1

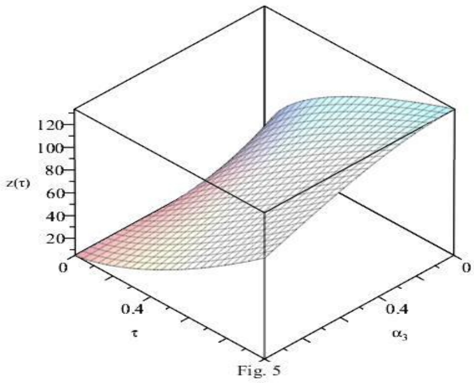

Figure-5: Variation of $z(\tau)$ for various values of α_3

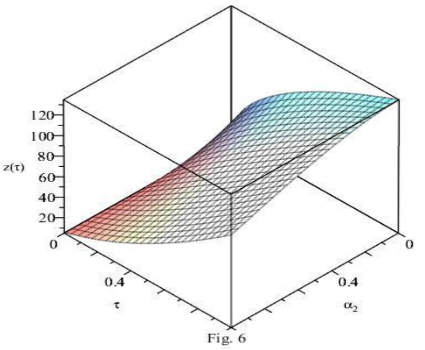

Figure-6: Variation of $z(\tau)$ for various values of α_2

5. CONCLUSIONS

In this article, Elzaki decomposition method has been applied for solving the fractional epidemic model. An excellent study between the previously distributed outcomes and current study have also been discussed through tables. Efficiency of the present study also determined through 3D-graphs.

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