On Applications of Generalized Fractional Calculus Operators

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Abstract: **In this paper we apply generalized fractional calculus operators on extended generalized hypergeometric function involving Srivastava extended beta function and establish some interesting results in terms of Hadamard products. Various integral transformations of these operators have also been established.**

Keywords: **Saigo-Maeda Fractional integral operators, Hadamard Product, Extended-Hypergeometric function, Srivastava Extended- Beta unction, Pathway Transform**

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1. INTRODUCTION

Fractional calculus has considerable importance in vast areas of applied sciences. Uses of Fractional calculus in various areas have extensively been studied by several researchers [1-4]. The exhaustive literature can also be found in the works [5-8]

In 1996, Saigo M. and Maeda [9] introduced and defined the Saigo Maeda generalized Fractional integral operators as

$$\left(I_{0,+}^{\zeta,\zeta',\xi,\xi',\mu} f\right)(x)=\frac{x^{-\zeta}}{\Gamma(\mu)}$$

$$.\int_0^x (x-v)^{\mu-1}v^{-\zeta'}F_3(\zeta,\zeta',\xi,\xi';\mu;1-\frac{v}{x},1-\frac{x}{v})f(v)dv \quad \ldots(1)$$

$$\left(I_{-}^{\zeta,\zeta',\xi,\xi',\mu} f\right)(x)=\frac{x^{-\zeta'}}{\Gamma(\mu)}$$

$$.\int_x^\infty (v-x)^{\mu-1}v^{-\zeta}F_3(\zeta,\zeta',\xi,\xi';\mu;1-\frac{x}{v},1-\frac{v}{x})f(v)dv \quad \ldots(2)$$

where $\zeta,\zeta',\xi,\xi',\mu\in C; x\in R^+, \mathrm{Re}(\mu)>0$

The corresponding generalized fractional differential operators are

$$\left(D_{0,+}^{\zeta,\zeta',\xi,\xi',\eta} f\right)(x)=$$

$$\left(\frac{d}{dx}\right)^{[\mathrm{Re}(\eta)]+1}\left(I_{0,+}^{-\zeta',-\zeta,-\xi'+[\mathrm{Re}(\eta)]+1,-\xi,-\eta+[\mathrm{Re}(\eta)]+1} f\right)(x)$$

And

$$\left(D_{-}^{\zeta,\zeta',\xi,\xi',\eta} f\right)(x)=$$

$$\left(-\frac{d}{dx}\right)^{[\mathrm{Re}(\eta)]+1}\left(I_{-}^{-\zeta',-\zeta,-\xi'+[\mathrm{Re}(\eta)]+1,-\xi,-\eta+[\mathrm{Re}(\eta)]+1} f\right)(x)$$

where $\zeta,\zeta',\xi,\xi',\eta\in C; x\in R^+, \mathrm{Re}(\eta)>0$

For the present study we shall require the lemma [9] which is as follows:

Lemma 1: Let $\zeta,\zeta',\xi,\xi',\eta,\lambda\in C; \mathrm{Re}(\eta)>0$

(a) If $\max.\{\mathrm{Re}(\zeta+\zeta'+\xi-\eta), \mathrm{Re}(\zeta'-\xi'),0\}<\mathrm{Re}(\lambda)$, then

$$\left(I_{0,+}^{\zeta,\zeta',\xi,\xi',\eta}t^{\lambda-1}\right)(x)=$$

$$\frac{\Gamma(\lambda)\Gamma(-\zeta'+\xi'+\lambda)\Gamma(-\zeta-\zeta'-\xi+\eta+\lambda)}{\Gamma(\xi'+\lambda)\Gamma(-\zeta-\zeta'+\eta+\lambda)\Gamma(-\zeta'-\xi+\eta+\lambda)}x^{-\zeta-\zeta'-1+\eta+\lambda} \quad \ldots(3)$$

(b) If $1+\min\{\mathrm{Re}(\zeta+\xi'-\eta), \mathrm{Re}(\zeta+\zeta'-\eta),\mathrm{Re}(-\xi)\}>\mathrm{Re}(\lambda)$, then

$$\left(I_{-}^{\zeta,\zeta',\xi,\xi',\eta}t^{\lambda-1}\right)(x)=$$

$$\frac{\Gamma(-\xi+\lambda)\Gamma(\zeta+\zeta'-\eta+\lambda)\Gamma(\zeta+\xi'-\eta+\lambda)}{\Gamma(\lambda)\Gamma(\zeta-\xi+\lambda)\Gamma(\zeta+\zeta'+\xi'-\eta+\lambda)}x^{-\zeta-\zeta'+\eta-\lambda}$$

(4)

Lemma 2: Let $\zeta,\zeta',\xi,\xi',\eta,\lambda\in C$

(a) If $\mathrm{Re}(\lambda)>\max\{0,\mathrm{Re}(-\zeta+\xi),\mathrm{Re}(-\zeta-\zeta'-\xi'+\eta)\}$, then

$$\left(D_{0,+}^{\zeta,\zeta',\xi,\xi',\eta}t^{\lambda-1}\right)(x)= \frac{\Gamma(\lambda)\Gamma(-\xi+\zeta+\lambda)\Gamma(\zeta+\zeta'+\xi'-\eta+\lambda)}{\Gamma(-\xi+\lambda)\Gamma(\zeta+\zeta'-\eta+\lambda)\Gamma(\zeta+\xi'-\eta+\lambda)}x^{\zeta+\zeta'-\eta+\lambda-1} \quad (5)$$

(b) If $\mathrm{Re}(\lambda)>\max\{\mathrm{Re}(-\xi'),\mathrm{Re}(\zeta'+\xi-\eta), \mathrm{Re}(\zeta+\zeta'-\eta)+[\mathrm{Re}(\eta)]+1\}$, then

$$\left(D_{-}^{\zeta,\zeta',\xi,\xi',\eta}t^{-\lambda}\right)(x)= \frac{\Gamma(\xi'+\lambda)\Gamma(-\zeta-\zeta'+\eta+\lambda)\Gamma(-\zeta'-\xi+\eta+\lambda)}{\Gamma(\lambda)\Gamma(-\zeta'+\xi'+\lambda)\Gamma(-\zeta-\zeta'-\xi+\eta+\lambda)}x^{\zeta+\zeta'-\eta-\lambda} \quad (6)$$

On setting $\zeta'=0$, Saigo Maeda generalized fractional integral operators reduces to the well-known Saigo fractional integral operators [9] as follows:

$$\left(I_{0,+}^{\zeta,0,\xi,\xi',\eta} f\right)(x)=\left(I_{0,+}^{\eta,\zeta-\eta,-\xi} f\right)(x)$$

$$\left(I_{-}^{\zeta,0,\xi,\xi',\eta} f\right)(x)=\left(I_{-}^{\eta,\zeta-\eta,-\xi} f\right)(x)$$

Also, on

setting $\zeta = 0$, Saigo Maeda Fractional differential operators reduce to the following Saigo Differential operators:

$$\left(D_{0,+}^{0,\zeta',\xi,\xi',\eta} f\right)(x) = D\left(I_{0,+}^{\eta,\zeta'-\eta,\xi'-\eta} f\right)(x)$$

$$\left(D_{-}^{0,\zeta',\xi,\xi',\eta} f\right)(x) = \left(D_{-}^{\eta,\zeta'-\eta,\xi'-\eta} f\right)(x)$$

These fractional integral operators introduced by Saigo are interesting generalizations of several fractional integral operators like "Weyl, Riemann-Liouville, Erdélyi- Kober". The details can be found in the work by Kilbas et al [10]

The two-parameter extension of extended Beta function due to Srivastava et al [11] is

$$\mathrm{B}_{b,k}^{(k_l)}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(k_l; -\frac{b}{t} - \frac{k}{1-t}\right)dt$$

$$(\min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta)\} > 0; \min\{\mathrm{Re}(b), \mathrm{Re}(d)\} > 0)$$

(7)

In 2013, Luo and Raina [12] introduced and defined the following generalized hypergeometric function using the two-parameter extension of extended Beta function due to Srivastava et al [11]:

$$ {}_pF_q^{(k_l)}\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z; \alpha, \beta\right]$$

$$= \begin{cases} \sum_{m=0}^{\infty} (a_1)_m \prod_{j=1}^{q} \frac{\mathrm{B}_{\alpha,\beta}^{(k_l)}(a_{j+1}+m, b_j - a_{j+1})}{B(a_{j+1}, b_j - a_{j+1})} \frac{z^m}{m!} \\ \quad \text{for } (|z|<1; p = q+1) \text{ and } R(b_j) > R(a_{j+1}) > 0 \\ \sum_{m=0}^{\infty} \prod_{j=1}^{q} \frac{\mathrm{B}_{\alpha,\beta}^{(k_l)}(a_j+m, b_j - a_j)}{B(a_j, b_j - a_j)} \frac{z^m}{m!} \\ \quad \text{for } (z \in \mathrm{C}; p = q) \text{ and } R(b_j) > R(a_j) > 0 \\ \sum_{m=0}^{\infty} \prod_{i=1}^{r} \frac{1}{(b_i)_m} \prod_{j=1}^{p} \frac{\mathrm{B}_{\alpha,\beta}^{(k_l)}(a_j+m, b_{r+j} - a_j)}{B(a_j, b_{r+j} - a_j)} \frac{z^m}{m!} \\ \quad \text{for } (z \in \mathrm{C}; r = q-p, p<q) \text{ and } R(b_{r+j}) > R(a_j) > 0 \end{cases}$$

(8)

In our investigation we will use the Hadamard product applied on two analytic functions. The Hadamard product decomposes a known function in two unknown functions. If one of the given power series exhibits an entire function, then the Hadamard product also exhibits an entire function.

Considering $F(z) := \sum_{m=0}^{\infty} A_m z^m \quad \text{for } |z| < R_F$

(9)

and $G(z) := \sum_{m=0}^{\infty} B_m z^m \quad \text{for } |z| < R_G$ (10)

Here R_F and R_G are the radii of convergence of F(z) and G (z) respectively.

Hadamard product series of *F* (z) and *G* (z) is defined as

$$\{F * G\}(z) := \sum_{l=0}^{\infty} A_l B_l z^l = \{G * F\}(z) \quad \text{for } |z| < \Upsilon$$

…(11)

here

$$\Upsilon = \lim_{l\to\infty}\left|\frac{A_l B_l}{A_{l+1}B_{l+1}}\right| = \left(\lim_{l\to\infty}\left|\frac{A_l}{A_{l+1}}\right|\right)\left(\lim_{l\to\infty}\left|\frac{B_l}{B_{l+1}}\right|\right) = R_F . R_G$$

For more details pertaining to Hadamard product, one may refer to [13, 14]

In recent years, Kumar and co-workers [15-17] have applied Tsallis Statistics and Pathway Model and in numerous fields like thermos nuclear reaction rate theory in applied analysis and in astrophysics.

The pathway model, introduced by Mathai [18], is based on the principle of switching among generalized extended type1 beta family, type2 beta family and gamma family. For the variable pathway parameter following three functional forms of the pathway model arises

$$f(x) = \begin{cases} A|x|^{\gamma}[1 - p(1-\alpha)|x|^{\delta}]^{\frac{\lambda}{1-\alpha}}; \\ \quad \text{for } 1 - p(1-\alpha)|x|^{\delta} > 0, \alpha < 1 \\ B|x|^{\gamma}[1 + p(\alpha-1)|x|^{\delta}]^{-\frac{\lambda}{1-\alpha}}; \\ \quad \text{for } -\infty < x < \infty; \alpha > 1 \\ C|x|^{\gamma} e^{-p\lambda|x|^{\delta}}; \\ \quad \text{for } -\infty < x < \infty; \alpha \to 1 \end{cases}$$

(12)

where $p > 0, \delta > 0, \upsilon > 0, \lambda > 0$; A_1, A_2 and A_3 are the normalized constants.

Kumar [15] introduced and defined a fractional type of pathway transform as

$$\left(P_{\lambda'}^{\alpha,\beta,\gamma} F\right)(z) = \int_0^{\infty} D_{\alpha,\beta}^{\lambda',\gamma}(tz)F(t)dt \; ; \text{ for } x > 0$$

………………………………………………….(13)

where

$$D_{\alpha,\beta}^{\lambda',\lambda}(x) = \begin{cases} \int_0^{\left[\frac{1}{a(1-\lambda)}\right]^{1/\alpha}} u^{\eta-1}\,[\,1 - a(1-\lambda)\,u^{\alpha}\,]^{\frac{1}{1-\lambda}} e^{-x\,u^{-\beta}} du \\ \quad \text{for } x > 0; \eta \in C; a, \alpha, \beta > 0; \lambda < 1 \\ \int_0^{\infty} u^{\eta-1}[1 + a(\lambda-1)u^{\alpha}]^{-\frac{1}{1-\lambda}} e^{-xu^{-\beta}} du \\ \quad \text{for } x > 0; \eta \in C; a, \beta > 0; \alpha \in R; \lambda > 1 \end{cases}$$

…(14)

In the same paper Kumar defined that if *f (x)* be a function of real variable *x*, integrable over finite interval (a, b) where $x\in(a, b)$, $a>0$, then $\exists\ \eta$, $\eta\in R$ such that

i. For every arbitrary $b>0$, $\int_b^\tau e^{-\eta t} f(x)\,dx \to$ a finite value when $\tau\to\infty$

ii. For any arbitrary $a>0$,

$$\int_\vartheta^a [f(x)]dx \to \text{a finite value as } \vartheta\to 0^+$$

then the Pathway transform of $f(x)$ exists and is denoted by $P_\delta[f(x):p]$ for

$$\mathbb{R}\left(\frac{\ln[1+(\alpha-1)p]}{\alpha-1}\right)>\lambda' \quad \text{for } p\in C.$$

Here it is very important to note that as α goes to 1, the P_δ- transform leads to well-known Laplace transform i.e. $\lim_{\alpha\to1} P_\delta\{f(x):s\}=L\{f(x):s\}$

The Euler-Beta transform [19] of a complex values function $f(x)$, where x is real, is given by

$$B\{f(x);m,n\}=\int_0^1 x^{m-1}(1-x)^{n-1} f(x)dx;\ R(x)>0;R(m),R(n)>0$$

…(15)

2. MAIN RESULTS

Theorem 1: -

If $\zeta,\zeta',\ell,\ell',k,\chi\in C;R(b_j)>R(a_{j+1})>0$, $p=q+1, j=1,2,...,q;\ R(k)>0,\left|\frac{bt}{a}\right|<1$ with $R(\chi)>\max\{0,\mathrm{Re}(\zeta+\zeta'+\ell-k),\mathrm{Re}(\zeta'-\ell')\}$ then for $x>$ o

$$\left\{I_{0+}^{\zeta,\zeta',1,1',k}\left(t^{\chi-1}(a-bt)^r\cdot{}_pF_q^{(k_l)}\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};tz;\alpha,\beta\right)\right)\right\}(x)$$

$$=a^{-r}x^{\chi+k-\zeta-\zeta'-1}\sum_{m=0}^{\infty}\frac{(r)_m}{m!}\left(\frac{bx}{a}\right)^m\cdot{}_pF_q^{(k_l)}\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};xz;\alpha,\beta\right)*{}_3\Psi_3\left(xz\left|\begin{matrix}(\chi+m,1),\ (\chi+k+m-\zeta-\zeta'-\ell,1)\ (\chi+m+\ell'-\zeta',1)\\ (\chi+m+\ell',1),\ (\chi+m+k-\zeta-\zeta',1)\ (\chi+m+k-\zeta'-\ell,1)\end{matrix}\right.\right)$$

Proof:

To prove the theorem, we first translate the extended generalized hyper geometric function in series form using equation (8) and then reversing the sequence of fractional integral operator, integration and summations which is allowed under the conditions stated. Then by using the result (3) we get the desired outcome with a few minor simplifications using equation (11).

Theorem 2:-

If $\zeta,\zeta',\ell,\ell',k,\chi\in C;R(b_j)>R(a_{j+1})>0$, $p=q+1, j=1,2,...,q;\ R(k)>0,\left|\frac{bt}{a}\right|<1$ with $R(\chi)<1+\min\{R(-\ell),\mathrm{Re}(\zeta+\zeta'-k),\mathrm{Re}(\zeta+\ell'-k)\}$ then for $x>$ o

$$\left\{I_{0^-}^{\zeta,\zeta',1,1',k}\left(t^{\chi-1}(a-bt)^r\ {}_pF_q^{(k_l)}\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};\frac{z}{t};\alpha,\beta\right)\right)\right\}(x)$$

$$=a^{-r}x^{\chi+k-\zeta-\zeta'-1}\sum_{m=0}^{\infty}\frac{(r)_m}{m!}\left(\frac{bx}{a}\right)^m\cdot{}_pF_q^{(k_l)}\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};\frac{z}{x};\alpha,\beta\right)*{}_3\Psi_3\left(\frac{z}{x}\left|\begin{matrix}(1-\chi-m-1,1),\ (1-\chi-m+\zeta+\zeta'-k,\ 1),\ (1-\chi-m-\zeta-k+1',1)\\ (1-\chi-m,1),\ (1-\chi-m+\zeta+\zeta'-k+1+1',1),\ (1-\chi-m-1+\zeta,\ 1)\end{matrix}\right.\right)$$

Proof:

In order to prove theorem under the stated conditions we first follow the same guideline as followed to prove theorem 1 and then using the result (4) we arrive at the desired result after a little simplification using equation (11).

Theorem 3: -

If $\zeta,\zeta',\ell,\ell',k,\chi\in C;R(b_j)>R(a_{j+1})>0$, $p=q+1, j=1,2,...,q;R(k)>0,\left|\frac{bt}{a}\right|<1$ with $R(\chi)>\max\{0,\mathrm{Re}(\zeta+\zeta'+\ell-k),\mathrm{Re}(\zeta'-\ell')\}$; $R(\lambda)>0,R(\eta)>0$ then for $x>$ o

$$\mathrm{B}\left\{I_{0+}^{\zeta,\zeta',1,1',k}\left(t^{\chi-1}(a-bt)^r\cdot{}_pF_q^{(k_l)}\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};tz;\alpha,\beta\right)\right)(x);\lambda,\eta\right\}$$

$$=\Gamma(\eta)a^{-r}x^{\chi+k-\zeta-\zeta'-1}\sum_{m=0}^{\infty}\frac{(r)_m}{m!}\left(\frac{bx}{a}\right)^m\cdot{}_pF_q^{(k_l)}\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};x;\alpha,\beta\right)*{}_4\Psi_4\left(x\left|\begin{matrix}(\chi+m,1),\ (\chi+m-\zeta-\zeta'-1+k,1),\ (\chi+m+1'-\zeta',1),\ (\lambda,1)\\ (\chi+m+1',1),\ (\chi+m+k-\zeta-\zeta',1),\ (\chi+m+k-\zeta'-1,1),\ (\lambda+\eta,1)\end{matrix}\right.\right)$$

Proof:

In order to prove theorem 3 under the stated conditions we use the result (15) along with the same process followed to prove theorem 1 and theorem 2, we arrive at the desired outcome with a few simplifications using equation (11).

Theorem 4: -

If $\zeta, \zeta', \ell, \ell', k, \chi \in C$; $\delta > 1$; $R(b_j) > R(a_{j+1}) > 0$, $p = q+1, j = 1,2,...,q$; $R(k) > 0, \left|\frac{bt}{a}\right| < 1$ with $R(\chi) > \max\{0, \mathrm{Re}(\zeta + \zeta' + \ell - k), \mathrm{Re}(\zeta' - \ell')\}$ then for $x > o$

$$P_\delta\left[z^{\lambda-1}\left\{I_{0+}^{\zeta,\zeta',1,1',k}\left(\begin{matrix} t^{\chi-1}(a-bt)^r \\ \cdot\, {}_pF_q^{(k_l)}\left(\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}; tz; \alpha, \beta\right)\end{matrix}\right)(x)\right\}\right]$$

$$= a^{-r} x^{\chi+k-\zeta-\zeta'+1} \sum_{m=0}^{\infty} \frac{(r)_m}{m!}\left(\frac{bx}{a}\right)^m$$

$$\cdot\, {}_pF_q^{(k_l)}\left(\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}; x.\Omega(\delta; s); \alpha, \beta\right) * {}_4\Psi_3\left(x.\Omega(\delta; s)\left|\begin{matrix} (0,1), (\chi+m,1), \\ (\chi+m+1',1), \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (\chi+m+k-\zeta-\zeta'-1,1), \ (\chi+m+1'-\zeta',1) \\ (\chi+m+k-\zeta-\zeta',1), \ (\chi+m+k-\zeta'-1,1) \end{matrix}\right)$$

where $\Omega(\delta; s) = \left\{\dfrac{\delta - 1}{\ln\{1 + (\delta-1)\delta\}}\right\}$

Proof:

In order to prove Theorem 4, we use the result (13) along with the same process followed to prove theorem 1 and theorem 2, we arrive at the desired outcome with a few simplifications using equation (11).

Theorem 5: -

If $\zeta, \zeta', \ell, \ell', k, \chi \in C$; $R(b_j) > R(a_{j+1}) > 0$, $p = q+1, j = 1,2,...,q$; $R(k) > 0, \left|\frac{bt}{a}\right| < 1$ with $R(\chi) > \max\{0, \mathrm{Re}(-\zeta - \zeta' - \ell' + k), \mathrm{Re}(-\zeta + \ell)\}$ then for $x > o$

$$\left\{D_{0+}^{\zeta,\zeta',1,1',k}\left(t^{\chi-1}(a-bt)^r\ {}_pF_q^{(k_l)}\left(\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}; tz; \alpha, \beta\right)\right)\right\}(x)$$

$$= a^{-r} x^{\chi+\zeta+\zeta'-k-1} \sum_{m=0}^{\infty} \frac{(r)_m}{m!}\left(\frac{bx}{a}\right)^m {}_pF_q^{(k_l)}\left(\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}; xz; \alpha, \beta\right) * {}_3\Psi_3\left(xz\left|\begin{matrix} (\chi+m,1), \\ (\chi+m-1,1), \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (\chi+m+\zeta+\zeta'+1'-k,1) \ \ (\chi+m-1+\zeta,1) \\ (\chi+m+\zeta+\zeta'-k,1) \ \ (\chi+m+\zeta+1'-k,1) \end{matrix}\right)$$

Proof:

In order to prove Theorem 5, we use the result (5) along with the same process followed to prove theorem 1 and theorem 2, we arrive at the desired outcome with a few simplifications using equation (11).

3. CONCLUSION

The theorems derived here are quite broad in nature and several new findings can be determined from the results obtained here. The broad nature of the functions used in the theorem makes the Theorems suitable for many extensions.

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