A Study of Unified Integrals Involving the Generalized Legendre's Associated Function, the generalized Polynomial Set and $\overline{\mathrm{H}}$- Function with Applications

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Abstract: The aim of this paper is to evaluate three finite integrals involving the product of generalized Legendre associated function, generalized polynomial set and $\bar{H}$-function proposed by Inayat Hussain (1987) [6] which contain a certain class of Feynman integrals. Next, we establish three theorems as an application of our main findings and using three results of Orr and Bailey recorded in the well-known text by Slater [11]. Further, we evaluate certain new integrals by the applications of these Theorems, which are of interest themselves and sufficiently general in nature. Several new and known results follow as their special cases. For the sake of illustration, some special cases are mentioned briefly.

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1. INTRODUCTION:

The $\overline{\mathrm{H}}$-function occurring in the present paper was introduced by Inayat Hussain [6] and later studied by Buschman and Srivastava [1] and several others.

The $\overline{\mathrm{H}}$-function will be defined and represented in the following manner:

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left(z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right)$$

$$=\frac{1}{2\pi i}\int_L \bar{\phi}(\xi) z^{\xi} d\xi \qquad (z\neq 0) \qquad (1)$$

where

$$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (2)$$

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{\mathrm{H}}$- Function given by (1) have been given by Gupta, Jain and Agrawal [3]:

(i) $|\arg(z)| < 1/2\,\Omega\pi$ and $\Omega > 0$

(ii) $|\arg(z)| = 1/2\,\Omega\pi$ and $\Omega \geq 0$ (3)

and (a) $\mu \neq 0$ and the contour L is so chosen that $(c\mu+\lambda+1)<0$

(b) $\mu = 0$ and $(\lambda+1)<0$

$$\Omega=\sum_{1}^{M}\beta_j+\sum_{1}^{N}\alpha_jA_j-\sum_{M+1}^{Q}\beta_jB_j-\sum_{N+1}^{P}\alpha_j \qquad (4)$$

$$\mu=\sum_{1}^{N}\alpha_jA_j+\sum_{N+1}^{P}\alpha_j-\sum_{1}^{M}\beta_j-\sum_{M+1}^{Q}\beta_jB_j \qquad (5)$$

$$\lambda=\mathrm{Re}\left(\sum_{1}^{M}b_j+\sum_{M+1}^{Q}b_jB_j-\sum_{1}^{N}a_jA_j-\sum_{N+1}^{P}a_j\right)$$

$$+\frac{1}{2}\left(-M-\sum_{M+1}^{Q}B_j+\sum_{1}^{N}A_j+P-N\right) \qquad (6)$$

It may be noted that the conditions of validity given above are more general than those given earlier [1].

We give below few particular cases of the $\overline{\mathrm{H}}$-function which are not the special cases of Fox's H-function.

(i) $$g_1=(-1)^p g(e,\eta,f,p;z)$$

$$=\frac{\Gamma(p+1)\Gamma\left(\frac{1+f}{2}\right)}{\pi^{d/2}2^{p+d}\Gamma\left(\frac{d-1}{2}\right)\Gamma(e)\Gamma\left(e-\frac{f}{2}\right)}$$

$$\overline{H}_{3,3}^{1,3}\left[-z\left|\begin{matrix}(1-e,1;1),\left(1-e+\frac{f}{2},1;1\right),(1-\eta,1;1+p)\\(0,1),(-\frac{f}{2},1;1),(-\eta,1;1+p)\end{matrix}\right.\right] \tag{7}$$

The above function is connected with a certain class of Feynman integrals [6, p. 41211, Eq. 5].

Further if we take e=1+f/2 in eq. (7), we have:

$$g_1\left(1+\frac{f}{2},\eta,f,p;z\right)$$

$$=\frac{\Gamma(p+1)\Gamma\left(\frac{1+f}{2}\right)}{\pi^{d/2}(-1)^p 2^{p+d}\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(1+\frac{f}{2}\right)}\phi(z,p+1,\eta) \tag{8}$$

where ϕ (z,p+1,η) is a generalized Riemann zeta function [2, p.27, 1.11 (1)].

(ii)
$$\beta'F(d;\varepsilon)=\frac{-1}{4\pi^{d/2}(1+\varepsilon)^2}$$

$$\overline{H}_{3,2}^{1,3}\left[-\frac{1}{(1+\varepsilon)^2}\left|\begin{matrix}(0,1;1),(0,1;1)(-1/2,1;d)\\(0,1),(-1,1;1+d)\end{matrix}\right.\right] \tag{9}$$

The above function is the exact partition of the Gaussian Model in statistical mechanics [6, p. 4127, (28)].

(iii) The polylogarithm of order p [2, p.30, 1.11(14)] is given by

$$F(z,p)=\sum_{n=1}^{\infty}\frac{z^a}{n^p}=\overline{H}_{2,2}^{1,2}\left[-z\left|\begin{matrix}(1,1;1),(1,1;p)\\(1,1),(0,1;p)\end{matrix}\right.\right] \tag{10}$$

For p =2, the above function reduces into Euler's dilogaritham [2, p.31, 1.11.1 (22)].

The generalized polynomial set is defined by the following Rodrigues type formula [9, p.64, Eq.(2.18)]

$$S_n^{\mu,\delta,\tau}\left[x;w,s,q,A,B,m,\xi,l\right]$$

$$=\left(Ax+B\right)^{-\mu}\left(1-\tau x^w\right)^{-\delta/\tau}$$

$$T_{\xi,l}^{m+n}\left[\left(Ax+B\right)^{\mu+qn}\left(1-\tau x^w\right)^{\delta/\tau+sn}\right] \tag{11}$$

with the differential operator

$$T_{k,l}=x^l\left[k+x\frac{d}{dx}\right]$$

The explicit series form of this generalized sequence of functions is given by [9, p.71, Eq. (2.3.4)]

$$S_n^{\mu,\delta,\tau}\left[x;w,s,q,A,B,m,\xi,l\right]$$

$$=B^{qn}x^{l(m+n)}\left(1-\tau x^l\right)^{sn}l^{m+n}$$

$$\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\sum_{j=0}^{m+n}\sum_{i=0}^{j}\frac{(-1)^j(-j)_i(\mu)_i(-\sigma)_i(-\mu-qn)_i}{\sigma!r!j!i!(1-\mu-j)_i}$$

$$\times\left(-\frac{\delta}{\tau}-sn\right)_\sigma\left(\frac{i+\xi+wr}{l}\right)_{m+n}\left(\frac{-\tau x^w}{1-\tau x^w}\right)^\sigma\left(\frac{Ax}{B}\right)^i \tag{12}$$

Some special cases of (12) are given by Raijada in tabular form [9, p.65]. We shall use the following special cases.

If we put A=1, B=0 in (12) and letting $\tau\to 0$ and use the well known results

$$\underset{\tau\to 0}{Lt}\left(1-\tau x^w\right)^{\delta/\tau}=\exp\left(-\delta x^r\right),$$

$$\underset{|b|\to\infty}{Lt}(b)_n\left(\frac{z}{b}\right)^n=z^n$$

Therein, we arrive at the following important polynomial set

$$S_n^{\mu,\delta,0}\left[x\right]=S_n^{\mu,\delta,0}\left[x;w,q,1,0,m,\xi,l\right]$$

$$=x^{qn+l(m+n)}l^{m+n}$$

$$\times\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\left(\frac{\mu+qn+\xi+wr}{l}\right)_{m+n}\left(\delta x^w\right)^\sigma}{r!\sigma!} \tag{13}$$

The generalized Legendre associated function $P_\gamma^{\alpha,\beta}(x)$ is defined [8, p. 560, Eq. (3)] and represent as follows:

$$P_\gamma^{\alpha,\beta}\left(x\right)=\frac{\left(1+x\right)^{\beta/2}}{\left(1-x\right)^{\alpha/2}\Gamma\left(1-\alpha\right)}$$

$${}_2F_1\left[\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2};\frac{1-x}{2}\\ 1-\alpha\end{matrix}\right] \tag{14}$$

where β and γ are unrestricted and α is non-positive integer.

If we put α=β in (14), $P_\gamma^{\alpha,\beta}(x)$ reduces to the associated Legendre function $P_\gamma^{\alpha}(x)$ [5, p. 999, Eq. (8.704)]. Further if we take α = β = 0 in (14), $P_\gamma^{\alpha,\beta}$ degenerates into well known Legendre polynomials [9, p. 166, Eq. 2].

The following result [7, p. 343, Eq. (38)] will be required in the sequel:

$$\int_{-1}^{+1}\left(1-x\right)^\rho\left(1+x\right)^\sigma P_\gamma^{\alpha,\beta}\left(x\right)dx$$

$$=\frac{2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}\Gamma\left(1+\rho-\frac{\alpha}{2}\right)\Gamma\left(1+\sigma+\frac{\beta}{2}\right)}{\Gamma\left(2+\rho+\sigma+\frac{\beta-\alpha}{2}\right)\Gamma\left(1-\alpha\right)}$$

$${}_3F_2\left[\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2},1+\rho-\frac{\alpha}{2}\\ 1-\alpha,2+\rho+\sigma+\frac{\beta-\alpha}{2}\end{matrix};1\right] \tag{15}$$

where α is a non-positive integer and

$$\mathrm{Re}\left(1+\rho-\alpha/2\right)>0,\mathrm{Re}\left(1+\sigma+\beta/2\right)>0.$$

First integral

$$\int_0^1\left(1-x\right)^{\alpha/2}\left(1+x\right)^{-\beta/2}\left(1-x^2\right)^{\rho-1}P_\gamma^{\alpha,\beta}\left(x\right)S_n^{\mu,\delta,0}\left[y\left(1-x^2\right)^k\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[z\left(1-x^2\right)^h\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$y^{R'+v\sigma}l^{m+n}\sqrt{\pi}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\delta^\sigma\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma!r!}$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\Gamma\left(1-\alpha+t\right)}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2}\left[z\left|\begin{matrix}\left(1-\rho-kv\sigma-kR',h;1\right)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\rho-kv\sigma-kR'-t,h;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ \left(1-\rho-kv\sigma-kR'-\frac{t}{2},h;1\right)\left(1-\rho-kv\sigma-kR'-\frac{t}{2}-\frac{1}{2},h;1\right)\end{matrix}\right] \tag{16}$$

where $R'=qn+l(m+n)$ .The integral is valid under following conditions

(i) $\alpha>0;h\geq0,k\geq0$ (not both zero simultaneously).

(ii) $\mathrm{Re}\left(\rho\right)+h\min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right]+1>0.$

iii) The $\overline{H}$- Function occurring in (16) satisfy conditions corresponding appropriately to those given by (3), (4), (5) and (6).

Second integral

$$\int_0^1 x^{\rho-1}\left(1-x\right)^{\sigma-1}\left(1+x\right)^{-\beta/2}P_\gamma^{\alpha,\beta}\left(x\right)S_n^{\mu,\delta,0}\left[yx^u\left(1-x\right)^v\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[zx^h(1-x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$=y^{R'}l^{m+n}\sqrt{\pi}\sum_{\eta=0}^{m+n}\sum_{r=0}^{\eta}\frac{(-\eta)_r\left(\frac{\mu+\xi+qn+wr}{l}\right)_{m+n}}{\eta!r!}$$

$$\left(\delta y^w\right)^\eta\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\Gamma(1-\alpha+t)}$$

$$\overline{H}_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}(1-\rho-uw\eta-uR',h;1)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\sigma+\frac{\alpha}{2}-vw\eta-vR'-t,k;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\\left\{1-\sigma-\rho-t+\frac{\alpha}{2}(u+v)(R'+w\eta),h+k;1\right\}\end{matrix}\right] \tag{17}$$

where $R'=qn+l(m+n)$ and

(i) $\alpha>0;u\geq0,v\geq0;h\geq0,k\geq0$ (not both zero simultaneously)

(ii) $\mathrm{Re}(\rho)+h\min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right]+1>0$

(iii) $\mathrm{Re}\left(\sigma-\frac{\alpha}{2}\right)+k\min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right]+1>0$

(iv) The $\overline{H}$-Function occurring in (17) satisfy conditions corresponding appropriately to those given by (3), (4), (5) and (6).

Third integral

$$\int_{-1}^{+1}(1-x)^\rho(1+x)^\sigma P_\gamma^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\left[y(1-x)^u(1+x)^v\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$=y^{R'}l^{m+n}\sum_{\eta=0}^{m+n}\sum_{r=0}^{\eta}\frac{(-\eta)_r\left(\frac{\mu+\xi+qn+wr}{l}\right)_{m+n}}{\eta!r!}\left(\delta y^w\right)^\eta 2^{(u+v)(R'+w\eta)}$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{t!\Gamma(1-\alpha+t)}2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}\overline{H}_{P+2,Q+1}^{M,N+2}$$

$$\left[z2^{h+k}\left|\begin{matrix}\left(-\sigma-\frac{\beta}{2}-vw\eta-vR',k;1\right)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(\frac{\alpha}{2}-\rho-uw\eta-uR'-t,h;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\\left\{-\sigma-\rho-t-(u+v)(R'+w\eta)-1,h+k;1\right\}\end{matrix}\right] \tag{18}$$

where

(i) $\alpha>0;u\geq0,v\geq0;h\geq0,k\geq0$ (not both zero simultaneously)

(ii) $\mathrm{Re}\left(1+\sigma+\frac{\beta}{2}\right)+k\min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right]>0$

(iii) $\mathrm{Re}\left(1+\rho-\frac{\alpha}{2}\right)+h\min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right]>0$

(iv) The $\overline{H}$-Function occurring in (18) satisfy conditions corresponding appropriately to those given by (3), (4), (5) and (6).

Proof: Proof of first integral

Expressing the generalized Legendre function in terms of ${}_2F_1$ with the help of (14), generalized polynomial set in its series form using (13) and $\overline{H}$-function in terms of Mellin Barnes type contour integral with the help of equation (1) in the left hand side of (16). Now changing the order of summation and integration (which is permissible under the condition stated) we get the following form of the integral (say Δ)

$$\Delta=y^{R'}l^{m+n}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma!r!}$$

$$\left(\delta y^v\right)^\sigma\frac{1}{\Gamma(1-\alpha)}\frac{1}{2\pi i}\int_L\overline{\theta}(s)z^s\int_0^1\left(1-x^2\right)^{\rho+kv\sigma+kR'+hs-1}$$

$${}_2F_1\left[\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2};\frac{1-x}{2}\\1-\alpha\end{matrix}\right]dx\,ds \tag{19}$$

On evaluating the x integral on the right hand side of (19) with the help of a known result [12, p.61, Eq. 2.16 (iii)] and expressing the function ${}_2F_1$ so obtained in terms of series and interchanging the order of summation and integration, (19) takes the following form after a little simplification.

$$\Delta = y^{R'} l^{m+n} \sum_{\sigma=0}^{m+n} \sum_{r=0}^{\sigma} \frac{(-\sigma)_r \left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma! r!}$$

$$\left(\delta y^v\right)^{\sigma} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{t!\,\Gamma(1-\alpha+t)}$$

$$\frac{1}{2\pi i}\int_L \bar{\theta}(s) z^s \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\rho+kv\sigma+kR'+hs+t\right)}{\Gamma\left(\rho+kv\sigma+kR'+hs+\frac{1}{2}\right)}$$

$$\frac{\left\{\Gamma 2\left(\rho+kv\sigma+kR'+hs\right)\right\}}{\left\{\Gamma 2\left(\rho+kv\sigma+kR'+hs+\frac{t}{2}\right)\right\}} ds$$

Now applying the well known duplication formula

$\Gamma\left(1/2\right)\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma\left(n+1/2\right)$ and

interpreting the resulting expression in terms of $\overline{H}$-function, we easily arrive at the desired result (16).

Proof of Second integral:

To establish (17) expressing the generalized Legendre function with the help of (14), generalized polynomial set $S_n^{\mu,\delta,\tau}[x]$ and $\overline{H}$-function with the help of equation (1) changing the order of summation and integration (which is permissible under the condition stated) to obtain a integral, which can be easily evaluated with a known result [12, p.60, eq. 2.16 (ii)], we get the desired result.

Proof of the third integral:

To evaluate the integral (18), we make use of the result given by equation (15) and proceed in a manner similar to that given in proofs of first and second integrals.

Theorems

Now we shall obtain three theorems with the help of our first integral given by (14) and the following three results recorded in the well known text by slater [11, p. 75, Th. I].

THEOREM 1

If

$$(1-x)^{a+b-c}\,{}_2F_1\left[2a,2b;2c;x\right] = \sum_{n'=0}^{\infty} a_{n'} x^{n'}$$

then

$$\int_0^1 (1-x)^{\alpha/2}(1+x)^{-\beta/2}\left(1-x^2\right)^{\rho-1} P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta}\left[y\left(1-x^2\right)^k\right]$$

$${}_2F_1\left[a,b;c+\frac{1}{2};x\right]{}_2F_1\left[c-a,c-b;c+\frac{1}{2};x\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix}\right.\right]dx$$

$$= y^{R'} l^{m+n}\sqrt{\pi} \sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma} \frac{(-\sigma)_r\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma! r!}\left(\delta y^v\right)^{\sigma}$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\,\Gamma(1-\alpha+t)}\sum_{n'=0}^{\infty}\frac{(c)_{n'}}{\left(c+\frac{1}{2}\right)_{n'}}a_{n'}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2}\left[z\left|\begin{matrix}(1-\rho-n'-kv\sigma-kR',h;1) \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\rho-n'-kv\sigma-kR'-t,h;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left\{1-\rho-n'-kv\sigma-kR'-\frac{t}{2},h;1\right\}\left\{1-\rho-n'-kv\sigma-kR'-\frac{t}{2}-\frac{1}{2},h;1\right\}\end{matrix}\right] \tag{20}$$

THEOREM 2

If

$${}_2F_1\left[a,b;c;x\right]{}_2F_1\left[a,b;d;x\right] = \sum_{n'=0}^{\infty} c_{n'} x^{n'}$$

then

$$\int_0^1 (1-x)^{\alpha/2}(1+x)^{-\beta/2}\left(1-x^2\right)^{\rho-1}$$

$$P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta}\left[y\left(1-x^2\right)^k\right]$$

$${}_4F_3\left[a,b;\frac{c}{2}+\frac{d}{2},\frac{c}{2}+\frac{b}{2}-\frac{1}{2};a+b,c,d;4x(1-x)\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[z\left(1-x^2\right)^h\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$= y^{R'} l^{m+n} \sqrt{\pi} \sum_{\sigma=0}^{m+n} \sum_{r=0}^{\sigma} \frac{(-\sigma)_r \left(\frac{\mu+\xi+qn+vr}{l} \right)_{m+n}}{\sigma! r!} \left(\delta y^v \right)^\sigma$$

$$\sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2} + 1 \right)_t \left(-\gamma - \frac{\alpha-\beta}{2} \right)_t}{2^t t! \Gamma(1-\alpha+t)} \sum_{n'=0}^{\infty} \frac{(c+d-1)_{n'} (c)_{n'}}{(a+b)_{n'}}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2} \left[z \left| \begin{matrix} (1-\rho-n'-kv\sigma-kR',h;1) \left(1-\rho-n'+\frac{\alpha}{2}-kv\sigma-kR'-t,h;1\right) (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q} \left\{1-\rho-n'-kv\sigma-kR'-\frac{t}{2},h;1\right\} \left\{1-\rho-n'-kv\sigma-kR'-\frac{t}{2}-\frac{1}{2},h;1\right\} \end{matrix} \right. \right]$$

(21)

THEOREM 3
If

then $${}_2F_1[a,b;c;x]\,{}_2F_1[a,b;d;x] = \sum_{n'=0}^{\infty} c_{n'} x^{n'}$$

$$\int_0^1 (1-x)^{\alpha/2} (1+x)^{-\beta/2} (1-x^2)^{\rho-1}$$

$$P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta}\left[y(1-x^2)^k \right]$$

$${}_4F_3\left[a,b,d,c-a; \frac{b}{2}+\frac{d}{2}, \frac{b}{2}+\frac{d}{2}+\frac{1}{2}, c; \frac{-x^2}{4(1-x)} \right]$$

$$\overline{H}_{P,Q}^{M,N} \left[z(1-x^2)^h \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right] dx$$

$$= y^{R'} l^{m+n} \sqrt{\pi} \sum_{\sigma=0}^{m+n} \sum_{r=0}^{\sigma} \frac{(-\sigma)_r \left(\frac{\mu+\xi+qn+vr}{l} \right)_{m+n}}{\sigma! r!} \left(\delta y^v \right)^\sigma$$

$$\sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2} + 1 \right)_t \left(-\gamma - \frac{\alpha-\beta}{2} \right)_t}{2^t t! \Gamma(1-\alpha+t)} \sum_{n'=0}^{\infty} \frac{(c)_{n'} (d)_{n'}}{(d+b)_{n'}}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2} \left[z \left| \begin{matrix} (1-\rho-n'-kv\sigma-kR',h;1) (1-\rho-n'-kv\sigma-kR'-t,h;1) (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q} \left\{1-\rho-kv\sigma-kR'-n'-\frac{t}{2},h;1\right\} \left\{1-\rho-kv\sigma-kR'-n'-\frac{t}{2}-\frac{1}{2},h;1\right\} \end{matrix} \right. \right]$$

(22)

where The $\overline{H}$- Function occurring in these theorems satisfy the conditions corresponding appropriately to those given by (3), (4), (5) and (6) and the conditions of validity of our first integral (14) also hold good.

Proof: To prove theorem 1 we consider the well known theorem given by Slater [11, p. 75, Th. I] and multiply the equation by

$$(1-x)^{-\alpha/2} (1+x)^{-\beta/2} (1-x^2)^{\rho-1} P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta,0}\left[y(1-x^2)^k \right]$$

$$\overline{H}_{P,Q}^{M,N} \left[z(1-x^2)^h \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right]$$

and integrating with respect to x from x = 0 to 1 and using result (14), we easily arrive the required result.

Theorem 2 and Theorem 3 can be proved on lines similar to that of theorem 1 by starting with the results [11, p.79, Th. VII, Th. VIII].

2. APPLICATIONS

(I) Taking c = a in our theorem 1, the value of $a_{n'}$ comes out to be equal to $b_{n'}$ and the result yields the following interesting integral

$$\int_0^1 (1-x)^{\alpha/2} (1+x)^{-\beta/2} (1-x^2)^{\rho-1} P_\gamma^{\alpha,\beta}(x)$$

$${}_2F_1\left[a,b; a+\frac{1}{2}; x \right] S_n^{\mu,\delta}\left[y(1-x^2)^k \right]$$

$$\overline{H}_{P,Q}^{M,N} \left[z(1-x^2)^h \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right] dx$$

$$= y^{R'} l^{m+n} \sqrt{\pi} \sum_{\sigma=0}^{m+n} \sum_{r=0}^{\sigma} \frac{(-\sigma)_r \left(\frac{\mu+\xi+qn+vr}{l} \right)_{m+n}}{\sigma! r!} \left(\delta y^v \right)^\sigma$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\,t!\,\Gamma(1-\alpha+t)}\sum_{n'=0}^{\infty}\frac{(a)_{n'}(b)_{n'}}{n'!\left(a+\frac{1}{2}\right)_{n'}}$$

$$\overline{H}^{M,N+2}_{P+2,Q+2}\left[z\left|\begin{matrix}(1-\rho-n'-kv\sigma-kR',h;1)\,(1-\rho-n'-kv\sigma-kR'-t,h;1)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\left\{1-\rho-n'-kv\sigma-kR'-\frac{t}{2},h;1\right\}\left\{1-\rho-n'-kv\sigma-kR'-\frac{t}{2}-\frac{1}{2},h;1\right\}\end{matrix}\right.\right]$$

(23)

Further on putting $b=a+1/2$ and $a=-e$ (a non negative integer) in (23) we have

$$\int_0^1(1-x)^{\alpha/2+e}(1+x)^{-\beta/2}\left(1-x^2\right)^{\rho-1}P_{\gamma}^{\alpha,\beta}(x)$$

$$S_n^{\mu,\delta}\left[y\left(1-x^2\right)^k\right]$$

$$\overline{H}^{M,N}_{P,Q}\left[z\left(1-x^2\right)^h\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$=y^{R'}l^{m+n}\sqrt{\pi}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma!\,r!}\left(\delta y^v\right)^{\sigma}$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\,t!\,\Gamma(1-\alpha+t)}\sum_{n'=0}^{\infty}\frac{(-e)_{n'}}{n'!}$$

$$\overline{H}^{M,N+2}_{P+2,Q+2}\left[z\left|\begin{matrix}(1-\rho-n'-kv\rho-kR',h;1)\,(1-\rho-n'-kv\sigma-kR'-t,h;1)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\left\{1-\rho-kv\sigma-kR'-n'+\frac{t}{2},h;1\right\},\left\{1-\rho-kv\sigma-kR'-n'+\frac{t}{2}-\frac{1}{2},h;1\right\}\end{matrix}\right.\right]$$

(24)

(II) On putting b = c = d in theorem 2, we easily get the following result

$$\int_0^1(1-x)^{-\alpha/2}(1+x)^{-\beta/2}\left(1-x^2\right)^{\rho-1}P_{\gamma}^{\alpha,\beta}(x)$$

$${}_2F_1\left[a,c-\frac{1}{2};a+c;4x(1-x)\right]S_n^{\mu,\delta}\left[y\left(1-x^2\right)^k\right]$$

$$\overline{H}^{M,N}_{P,Q}\left[z\left(1-x^2\right)^h\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$=y^{R'}l^{m+n}\sqrt{\pi}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma!\,r!}\left(\delta y^v\right)^{\sigma}$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\,t!\,\Gamma(1-\alpha+t)}\sum_{n'=0}^{\infty}\frac{(2c-1)_{n'}(2a)_{n'}}{(a+c)_{n'}\,n'!}$$

$$\overline{H}^{M,N+2}_{P+2,Q+2}\left[z\left|\begin{matrix}(1-\rho-n'-kv\sigma-kR',h;1)\,(1-\rho-n'-kv\sigma-kR'-t,k;1)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\left\{1-\rho-kv\sigma-kR'-n'+\frac{t}{2},h;1\right\}\left\{1-\rho-kv\sigma-kR'-n'-\frac{t}{2}-\frac{1}{2},h;1\right\}\end{matrix}\right.\right]$$

(25)

Further if we put $a=-e$ in (25), it reduces to following interesting integral

$$\int_0^1(1-x)^{-\alpha/2}(1+x)^{-\beta/2}\left(1-x^2\right)^{\rho-1}P_{\gamma}^{\alpha,\beta}(x)$$

$${}_2F_1\left[-e,c-\frac{1}{2};c-e;4x(1-x)\right]S_n^{\mu,\delta}\left[y\left(1-x^2\right)^k\right]$$

$$\overline{H}^{M,N}_{P,Q}\left[z\left(1-x^2\right)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$=y^{R'}l^{m+n}\sqrt{\pi}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma!\,r!}\left(\delta y^v\right)^{\sigma}$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\Gamma(1-\alpha+t)}\sum_{n'=0}^{\infty}\frac{(2c-1)_{n'}(-2e)_{n'}}{(c-e)_{n'}n'!}$$

$$\overline{H}^{M,N+2}_{P+2,Q+2}\left[z\left|\begin{matrix}(1-\rho-n'-kv\sigma-kR',h;1)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-n'-kv\sigma-vR',h;1)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\\left\{1-\rho-kv\sigma-n'-kR'+\frac{t}{2},h;1\right\}\left\{1-\rho-kv\sigma-n'-kR'-\frac{t}{2}-\frac{1}{2},h;1\right\}\end{matrix}\right]$$

(26)

If we put $b=c=d$ in theorem 3, we arrive at the following integral after a little simplification

If $(1-x)^{-2a}=\sum_{n'=0}^{\infty}d_{n'}z^{n'}$ then

$$\int_0^1(1-x)^{\alpha/2}(1+x)^{-\beta/2}\left(1-x^2\right)^{\rho-1}P_\gamma^{\alpha,\beta}(x)$$

$${}_2F_1\left[a,c-a;c+\frac{1}{2};-x^2/4(1-x)\right]S_n^{\mu,\delta}\left[y\left(1-x^2\right)^k\right]$$

$$\overline{H}^{M,N}_{P,Q}\left[z\left(1-x^2\right)^h\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$=y^{R'}l^{m+n}\sqrt{\pi}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma!r!}\left(\delta y^v\right)^\sigma$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\Gamma(1-\alpha+t)}\sum_{n'=0}^{\infty}\frac{(c)_{n'}(2a)_{n'}}{(2c)_{n'}n'!}$$

$$\overline{H}^{M,N+2}_{P+2,Q+2}\left[z\left|\begin{matrix}(1-\rho-n'-kv\sigma-kR',h;1)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-kv\sigma-kR'-n'-t,h;1)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\\left\{1-\rho-kv\sigma-n'-kR'-\frac{t}{2},h;1\right\}\left\{1-\rho-kv\sigma-n'-kR'-\frac{t}{2}-\frac{1}{2},h;1\right\}\end{matrix}\right]$$

(27)

Special Cases:

(I) If we reduce the $\overline{H}$-function occurring in theorem 1 to the g function given by (7), by suitably choosing the parameters, we get the following form

If

$$(1-x)^{a+b-c}\ {}_2F_1[2a,2b;2c;x]=\sum_{n'=0}^{\infty}a_{n'}z^{n'}$$

then

$$\int_0^1(1-x)^{\alpha/2}(1+x)^{-\beta/2}\left(1-x^2\right)^{\rho-1}P_\gamma^{\alpha,\beta}(x)$$

$${}_2F_1\left[a,b;c+\frac{1}{2};x\right]{}_2F_1\left[c-a,c-b;c+\frac{1}{2};x\right]$$

$$S_n^{\mu,\delta}\left[y\left(1-x^2\right)^k\right]g\left(e,\varsigma,f,\tau,z\left(1-x^2\right)^h\right)dx$$

$$=y^{R'}l^{m+n}\sqrt{\pi}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\frac{(-\sigma)_r\left(\frac{\mu+\xi+qn+vr}{l}\right)_{m+n}}{\sigma!r!}\left(\delta y^v\right)^\sigma$$

$$\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^{t+\tau+d}t!\Gamma(1-\alpha+t)}\sum_{n'=0}^{\infty}\frac{(c)_{n'}(2a)_{n'}}{(2c)_{n'}n'!}$$

$$\frac{(-1)^p\Gamma(p+1)\Gamma\left(\frac{1+f}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma(e)\Gamma\left(e-\frac{f}{2}\right)}$$

$$\overline{H}^{1,5}_{5,5}\left[-z\left|\begin{matrix}(1-\rho-n'-kv\sigma-kR',h;1)\\(0,1),(\frac{-\rho}{2},1;1)(-\eta,1;p+1)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-kv\sigma-kR'-n'-t,h;1)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\\left\{1-\rho-kv\sigma-n'-kR'-\frac{t}{2},h;1\right\}\left\{1-\rho-kv\sigma-n'-kR'-\frac{t}{2}-\frac{1}{2},h;1\right\}\end{matrix}\right]$$

(II) On putting $q=\xi=m=0$ and $l=-1$ in (14) the generalized polynomial set $S_n^{\mu,\delta,0}$ reduces to the class of polynomials studied by Gould and Hopper [4] we arrive at the following integral

$$\int_0^1 (1-x)^{\alpha/2}(1+x)^{-\beta/2}(1-x^2)^{\rho-1}$$

$$P_\gamma^{\alpha,\beta}(x) H_n^{(r)}\left[y(1-x^2)^k, \mu, \delta\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[z(1-x^2)^h \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] dx$$

$$= y^{-1}(-1)^n \sqrt{\pi} \sum_{\sigma=0}^{n}\sum_{r=0}^{\sigma} \frac{(-\sigma)_r (-vr)_n}{\sigma! r!}$$

$$\sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2}\left[z \left| \begin{matrix} (1-\rho-kv\sigma+kn,h;1) \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right.$$

$$(1-\rho-kv\sigma+kn-t,h;1)$$

$$\left(1-\rho-kv\sigma+kn-\frac{t}{2},h;1\right)$$

$$\left.\begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\rho-kv\sigma+kn-\frac{t}{2}-\frac{1}{2},h;1\right) \end{matrix}\right]$$

(III) In theorem 1 if we put ρ=1,α=β=0 and reduce the generalized associated legendre function $P_\gamma^{\alpha,\beta}(x)$ and generalized polynomial set $S_n^{\mu,\delta,0}(x)$ to unity and H-function to well known G function, we get a known result of Srivastava [13].

3. CONCLUSION

In this paper three integrals have been evaluated and then further used to establish three theorems. On account of generalization the functions present in the integrand are very general in nature and so one can reduce them in a large number of other simpler functions.

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