A Modified Homotopy Analysis Method to Solve Fractional Telegraph Equation

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Abstract: **The objective of this paper is to apply Homotopy analysis natural transform method for finding solution of telegraph equation of fractional order. This method is based on homotopy analysis method (HAM) which provides us with a simple yet effective way to adjust and control the convergence region of the infinite series solution by introducing the auxiliary parameter h. Fractional derivatives are described in Caputo sense. Numerical solutions are obtained to compare accuracy and ease of implementation.**

Keywords: **Fractional differential equations, Caputo type derivatives, modified homotopy analysis method, numerical solutions.**

1. INTRODUCTION

In recent years, a lot of research has been done on finding numerical solutions of fractional differential equations. In this context, many new methods have been investigated such as Adomian decomposition method [1-2], Homotopy perturbation method [3, 4], Homotopy analysis method (HAM) [5-10], Laplace Decomposition Method [11], Homotopy Perturbation Sumudu Transform Method [12] and Homotopy Analysis Transform Method [13] to mention a few.

One more variant of HAM is Homotopy Analysis Natural Transform Method (HANTM)which has been introduced by Ziane and Cherif [14] for the solution of nonlinear partial differential equations. This method is a combination of Natural transform method [15] and Homotopy analysis method. The natural transform is a transform similar to Laplace transform as well, as the Sumudu transform and is defined by an integral. This transform along with HAM has been used by many researchers in the resolution of linear differential equations [16-19].

In this paper we aim at solving fractional telegraph equation by the so called HANTM. The telegraph equation is a partial differential equation with constant coefficients which arises in the study of propagation of electrical signals in a cable of transmission line and has applications in numerous fields such as wave propagation [20], signal analysis [21], random walk theory [22], *etc.*

The paper is organized as follows. Some basic definitions and properties used throughout the paper are given in Section 2. In Section 3, we apply Homotopy Analysis Natural Transform Method for finding accurate approximate solutions to the fractional telegraph equation. Section 5 contains final conclusions.

2. BASIC DEFINITIONS

Here we present some basic properties and definitions of fractional calculus theory, natural transform and natural transform of fractional derivatives which are used further in this paper.

Fractional calculus. In the literature, many definitions of derivatives of non-integer order are available. We introduce here definition and properties of Riemann–Liouville fractional integration and definition of fractional differentiation given by Caputo.

Definition 2.1. The definition of Riemann-Liouville fractional integral of order α of a function $f(x)$ is given by [23]

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \alpha > 0, x > 0, \qquad (2.1)$$

$$I_x^{0} f(x) = f(x).$$

Definition 2.2.

The Caputo fractional derivative of $f(x)$ is defined as

$$^{c}D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt, & n-1 < \alpha < n \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n. \end{cases}$$

(2.2)

N-transform.

We give some basic definitions and properties of the N Transform which are used further in this paper (see [15], [19]).

Definition 2.3

The Natural Transform of the function $f(x)>0$ and $f(x)=0$ for $x<0$ is defined over the set of functions:

$$A = \left\{ f(x) \middle| \exists M, \tau_1, \tau_2 > 0, |f(x)| < Me^{\frac{|x|}{\tau_j}}, \text{ if } x \in (-1)^j \times [0, \infty) \right\}$$

by the following integral

$$\mathbb{N}^{+}\left[f(x)\right]=\eta(s,u)=\int_{0}^{\infty}e^{-sx}f(ux)dx; s>0, u>0 \quad (2.3)$$

and the inverse Natural Transform of the function $f(x)$ is defined by

$$\mathbb{N}^{-1}\{\eta(s,u)\}=f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\frac{sx}{u}}R(s,u)ds \quad (2.4)$$

where s and u are the Natural Transform variables.
Further if $\eta(s,u)$ is the N-Transform of the function, then we also have the following results

Definition 2.4, N-Transform of fractional integral of order α is defined by

$$\mathbb{N}^{+}\left[\left(I_{x}^{\alpha}f\right)(x)\right]=\frac{s^{\alpha}}{u^{\alpha}}\eta(s,u) \quad (2.5)$$

Definition 2.5 N-Transform of fractional derivative of order α is defined by

$$\mathbb{N}^{+}\left[\left({}^{c}D_{x}^{\alpha}f\right)(x)\right]=\frac{s^{\alpha}}{u^{\alpha}}\eta(s,u)-\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{k}(0) \quad (2.6)$$

Definition 2.6 N-Transform of $f(x)$ multiplied with shift function x^n is given by

$$\mathbb{N}^{+}\left[x^{n}f(x)\right]=\frac{s^{n}}{u^{n}}\frac{d^{n}}{du^{n}}u^{n}\eta(s,u) \quad (2.7)$$

and

$$\mathbb{N}^{+}\left[\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right]=\frac{u^{\alpha}}{s^{\alpha+1}}, \alpha\geq 0 \quad (2.8)$$

3. HOMOTOPY ANALYSIS NATURAL TRANSFORM METHOD

In this section we apply the HANTM to the following space-fractional telegraph equation:

$${}^{c}D_{x}^{\alpha}U(x,t)=\frac{\partial^{2}U(x,t)}{\partial t^{2}}+\frac{\partial U(x,t)}{\partial t}+U(x,t); t\geq 0, 1<\alpha\leq 2 \quad (3.1)$$

subject to the conditions

$$\begin{aligned} U(x,t)\big|_{x=0}&=e^{-t}, \quad t\geq 0,\\ U_{x}(x,t)\big|_{x=0}&=e^{-t}, \quad t\geq 0. \end{aligned} \quad (3.2)$$

where ${}^{c}D_{x}^{\alpha}$ is Caputo fractional derivative defined by (2.2).

Applying the N-Transform on both sides of telegraph equation (3.1), we get

$$\mathbb{N}^{+}\left[{}^{c}D_{x}^{\alpha}U(x,t)\right]=\mathbb{N}^{+}\left[\frac{\partial^{2}U(x,t)}{\partial t^{2}}+\frac{\partial U(x,t)}{\partial t}+U(x,t)\right] \quad (3.3)$$

using the property of N-transform, we have the following form

$$\mathbb{N}^{+}\left[U(x,t)\right]-\frac{u^{\alpha}}{s^{\alpha}}\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}U_{x}^{k}(0,t)-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left(U_{tt}+U_{t}+U\right)=0 \quad (3.4)$$

Using initial conditions given by (3.2), we further get

$$\mathbb{N}^{+}\left[U(x,t)\right]-\left(\frac{1}{s}+\frac{u}{s^{2}}\right)e^{-t}-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left(U_{tt}+U_{t}+U\right)=0 \quad (3.5)$$

We define the nonlinear part as

$$\mathcal{R}\left[\Phi(x,t,q)\right]=\mathbb{N}^{+}\left[\Phi\right]-\left(\frac{1}{s}+\frac{u}{s^{2}}\right)e^{-t}-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left(\Phi_{tt}+\Phi_{t}+\Phi\right) \quad (3.6)$$

By means of homotopy analysis method, we construct the so-called zeroth-order deformation equation

$$(1-q)\mathbb{N}^{+}\left[\Phi(x,t,q)-U_{0}(x,t)\right]=qhH(x,t)\mathcal{R}\left[\Phi(x,t,q)\right] \quad (3.7)$$

where $q\in[0,1]$ is the embedding parameter, $H(x,t)\neq 0$ is an auxiliary function, $h\neq 0$ is an auxiliary parameter, $\mathbb{N}^{+}$ is an auxiliary linear N-Transform operator and $\Phi(x,t;q)$ is an unknown function. When $q=0$ and $q=1$, we have

$$\begin{aligned}\Phi(x,t,0)&=U_{0}(x,t)\\ \Phi(x,t,1)&=U(x,t)\end{aligned} \quad (3.8)$$

When q increases from 0 to 1, $\Phi(x,t;q)$ varies continuously from the initial approximation $U_{0}(x,t)$ to the exact solution $U(x,t)$.

Using the parameter q, we expand $\Phi(x,t;q)$ in Taylor series as follows:

$$\Phi(x,t;q)=U_{0}(x,t)+\sum_{m=1}^{\infty}U_{m}(x,t)q^{m}, \quad (3.9)$$

where $U_{m}(x,t)=\frac{1}{m!}\frac{\partial^{m}\Phi(x,t;q)}{\partial q^{m}}\bigg|_{q=0}$. (3.10)

Now taking $q=1$ in (3.9) and using (3.8), we get

$$U(x,t)=U_0(x,t)+\sum_{m=1}^{\infty}U_m(x,t). \tag{3.11}$$

We define the vectors

$$\vec{U}_n=\{U_0(x,t),U_1(x,t),U_2(x,t),...,U_n(x,t)\}$$

Differentiating (3.7) *m* times with respect to *q*, then setting *q*=0, and finally dividing by *m*!, we have the so-called *m*th-order deformation equation

$$\mathbb{N}^{+}\left[U_m(x,t)-\chi_m U_{m-1}(x,t)\right]=h\mathcal{R}_m\left[U_{m-1}(x,t)\right], \tag{3.12}$$

where

$$\mathcal{R}_m\left(\vec{U}_{m-1}(x,t)\right)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathcal{R}\left[\Phi(x,t,q)\right]}{\partial q^{m-1}}\right|_{q=0}, \tag{3.13}$$

Applying the inverse N-Transform on both side of equation (3.12), we have

$$U_m(x,t)=\chi_m U_{m-1}(x,t)+\mathbb{N}^{-1}\left[h\mathcal{R}_m\left(\vec{U}_{m-1}(x,t)\right)\right] \tag{3.14}$$

From (3.14), we have

$$\begin{aligned}
U_1(x,t)&=h\mathbb{N}^{-1}\mathcal{R}_1\left[\vec{U}_0(x,t)\right]\\
U_2(x,t)&=U_1+h\mathbb{N}^{-1}\left[\mathcal{R}_2\vec{U}_1(x,t)\right]\\
U_3(x,t)&=U_2+h\mathbb{N}^{-1}\left[\mathcal{R}_3\vec{U}_2(x,t)\right]
\end{aligned} \tag{3.15}$$

Where

$$\begin{aligned}
\mathcal{R}_1\left[\vec{U}_0(x,t)\right]&=\mathbb{N}^{+}[U_0]-\left(\frac{1}{s}+\frac{u}{s^2}\right)e^{-t}\\
&\quad-\frac{u^\alpha}{s^\alpha}\mathbb{N}^{+}\left(U_{0tt}+U_{0t}+U_0\right)\\
\mathcal{R}_2\left[\vec{U}_1(x,t)\right]&=\mathbb{N}^{+}[U_1]-\frac{u^\alpha}{s^\alpha}\mathbb{N}^{+}\left(U_{1tt}+U_{1t}+U_1\right)\\
\mathcal{R}_3\left[\vec{U}_2(x,t)\right]&=\mathbb{N}^{+}[U_2]-\frac{u^\alpha}{s^\alpha}\mathbb{N}^{+}\left(U_{2tt}+U_{2t}+U_2\right)
\end{aligned} \tag{3.16}$$

Using the initial condition (3.2), the iteration formula (3.15) and (3.16), we obtain

$$\begin{aligned}
U_0(x,t)&=(1+x)e^{-t}\\
U_1(x,t)&=-he^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]\\
U_2(x,t)&=-h(1+h)e^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]+\\
&\quad h^2e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],\\
U_3(x,t)&=-h(1+h)^2e^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]\\
&\quad+2h^2(1+h)e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\\
&\quad-h^3e^{-t}\left[\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\right],\\
&\ \ \vdots
\end{aligned} \tag{3.17}$$

Proceeding similarly, we can find other components of HANTM.So approximate solution of (3.1) in a series form is

$$\begin{aligned}
U(x,t)&=e^{-t}(1+x)+(-h)\left(3+3h+h^2\right)e^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)}\right.\\
&\quad\left.+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]+\left(3h^2+2h^3\right)e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\
&\quad\left.+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]+\left(-h^3\right)e^{-t}\left[\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\right]+\ldots
\end{aligned} \tag{3.18}$$

When *h*=−1, the approximate solution of (3.1) is given by

$$\begin{aligned}
U(x,t)&=e^{-t}\left[1+x+\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\
&\quad\left.+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right]
\end{aligned} \tag{3.19}$$

and when α=2, we obtain

$$U(x,t)=e^{-t}\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\ldots\right]=e^{x-t} \tag{3.20}$$

which is an exact solution to the fractional telegraph equation .

4. NUMERICAL RESULTS

(a)

(b)

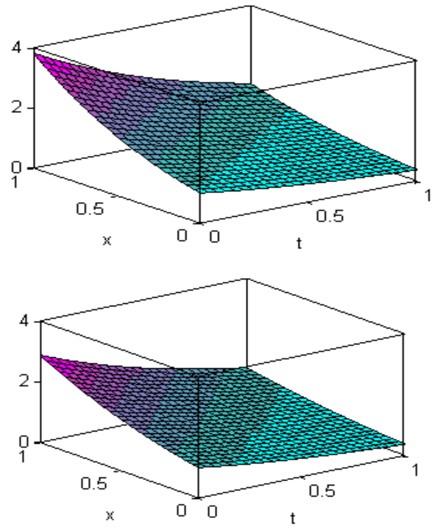

Figure 4.1: Surface showing approximation solution of equation (3.1) when (a) $\alpha = 1.25$, $h = -1.5$ (b) $\alpha = 1.75$, $h = -0.8$.

(a)

(b)

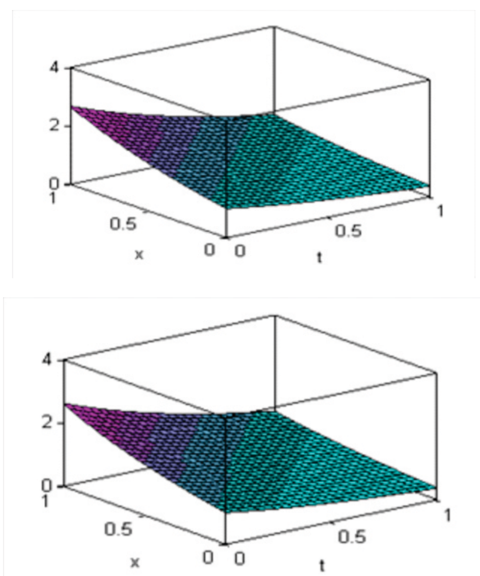

Figure 4.2: Surface showing the approximation solution of equation (3.1) when $\alpha = 2$ and (a) $h = -0.8$(b) $h = -1.5$.

5. CONCLUSION

The present work may be helpful in understanding HANTM, an efficient modification of HAM in which Natural transform is coupled with HAM. An application of the approach is made to solve fractional order telegraph equation. We have shown that approximate solutions obtained by HANTM are highly accurate. The graphs of the numerical results are also shown.

REFRENCES

[1.] S. Momani and K. Al-Khaled, Numerical solution for systems of fractional differential equation by the decomposition method, Applied Mathematics and Computation 162 (2005), 1351–1365.

[2.] N. T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, Appl. Math. Comput. 131 (2007), 517–529.

[3.] Q. Wang, Homotopy perturbation method for fractional order KdV equation, Applied Mathematics and Computation 190 (2007), no. 2, 1795–1802.

[4.] S. Momani and Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Physics Letters A 365 (2007), 345–350.

[5.] S. J. Liao, An approximate solution technique which does not depend upon small parameters: a special example, Int. J. Nonlinear Mech 30 (1995), 371–380.

[6] S. J. Liao, Analytic solution of the temperature distribution In Blasius viscous flow problems, Journal of Fluid Mechanics, 453 (2002), 411–425.

[7] S. J. Liao and K. Cheung, Homotopy analysis method of nonlinear progressive waves in deep water, Journal of Applied Engineering Mathematics 45 (2003), no. 2, 105–116.

[8] S. J. Liao, Comparison between the homotopy analysis method and Homotopy perturbation method, Applied Mathematics and Computation 169 (2005), 1186–1194.

[9] S. J. Liao, Homotopy analysis method; A new analytic method for nonlinear problems, Journal of Applied Mathematics and Mechanics, 19 (1998), 957–962.

[10] S. J. Liao, On the Homotopy analysis method for nonlinear problems, Applied Mathematics and Computation, 147 (2004), 499–513.

[11] Yasir Khan, An effective modification of the Laplace decomposition method for nonlinear equations, International Journal of Nonlinear Sciences and Numerical Simulation.,10 (2009), 1373-1376.

[12] Y. Khan, Q. Wu, Homotopy perturbation transform method for nonlinear equations using He's polynomials, Computer and Mathematics with Applications, 61 (2011) ,1963-1967.

[13] M. Khan, MA. Gondal, I. Hussain, S. Karimi Vanani, A new comparative study between homotopy analysis transform method and homotopy perturbation transform method on semi infinite domain, Mathematical and Computer Modelling., 55 (2012) ,1143-1150.

[14] D. Ziane and M. Hamdi cherif , modified homotopy analysis method for nonlinear fractional partial differential equations, international journal of analysis and applications volume 14, number 1 (2017), 77-87

[15] Z. H. Khan and W. A. Khan, N-transform properties and applications, NUST J. Eng. Sci, 1 (2008), 127-133.

[16] R. Silambarasn and F. B. M. Belgacem, Applications of the Natural Transform to Maxwell's Equations, Prog. Elect. Res. Symp. Proc. Suzhou. China, (2011), 899-902.

[17] S. K. Q. Al-Omari, On the Application of Natural Transforms, Int. J. Pure Appl. Math. 85 (2013), 729-744.

[18] M. Junaid, Application Of Natural Transform To Newtonian Fluid Problems, Int. J. Sci. Technol. 5 (2016), 138-147.

[19] D. Loonker and P. K. Banerji, Solution of Fractional Ordinary Differential Equations by Natural Transform, Int. J. Math. Eng. Sci. 2 (2013), 1-7.

[20] Weston VH, He S:Wave splitting of the telegraph equation in R3 and its application to inverse scattering. Inverse Probl. 1993, 9: 789–812.

[21] Jordan PM, Puri A: Digital signal propagation in dispersive media. J. Appl. Phys. 1999, 85: 1273–1282.

[22] Banasiak J, Mika R: Singular perturbed telegraph equations with applications in random walk theory. J. Appl. Math. Stoch. Anal. 1998, 11: 9–28.

[23] Miller, K.S. and Ross, B.,An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, 1993.

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